Indices and Surds

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Mathematical Terms, Diagnostic Test, Revision Assignment, Working Mathematically

Learning Outcomes
Students will be able to:
- Apply index laws to evaluate arithmetic expressions.
- Apply index laws to simplify algebraic expressions.
- Use standard (scientific) notation to write small and large numbers.
- Understand the difference between rational and irrational numbers.
- Perform operations with surds and indices.

Areas of Interaction
Approaches to Learning (Knowledge Acquisition, Logical Thinking, IT Skills, Reflection), Human Ingenuity
5:01 | Indices and the Index Laws

- 5 is called the base.
- 4 is called the index.
- 625 is the basic numeral.

\[ x^n = x \times x \times x \times \ldots \times x \times x \] (where \( n \) is a positive integer)

For:
- \( x \) is the base
- \( n \) is the index.

**Multiplication using indices**

- \( 5^3 \times 5^2 = (5 \times 5 \times 5) \times (5 \times 5) \)
- \( x^3 \times x^2 = (x \times x \times x \times x) \times (x \times x \times x) \)

- **Law 1** When multiplying terms, *add* the indices: \( x^m \times x^n = x^{m+n} \)

**Division using indices**

- \( 5^6 \div 5^3 = \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5 \times 5} \)
- \( x^4 \div x^3 = \frac{x \times x \times x \times x}{x \times x \times x} \)

- **Law 2** When dividing terms, *subtract* the indices: \( x^m \div x^n = x^{m-n} \)

**Powers of indices**

- \( (5^2)^3 = 5^2 \times 5^2 \times 5^2 \)  [Using Law 1]
- \( (x^3)^4 = x^5 \times x^3 \times x^5 \times x^5 \)  [Using Law 1]

- **Law 3** For powers of a power, *multiply* the indices: \( (x^m)^n = x^{mn} \)

If we simplify the division \( x^n \div x^n \), using the second law above:

\[ x^n \div x^n = x^{n-n} = x^0 \]

But any expression divided by itself must equal 1.

\[ x^n \div x^n = 1 \]

Therefore \( x^0 \) must be equal to 1.

- **Law 4** \( x^0 = 1 \)
worked examples

1 Simplify:
   a \(3^3\)
   b \(11^4\)
   c \((-3)^2\)

2 Simplify:
   a \(5^2 \times 5^4\)
   b \(a^2 \times a^4\)
   c \(3x^4 \times 5x^2y^2\)

3 Simplify:
   a \(a^3 + a\)
   b \(18x^6 + 6x^3\)
   c \((x^3)^4 + (x^2)^3\)

4 Simplify:
   a \((x^3)^3\)
   b \((3x^3)^4\)
   c \((2y^3)^4 + (4y^6)^2\)

5 Simplify:
   a \(7^0\)

Solutions

1 a \(3^3 = 3 \times 3 \times 3 = 27\)
   b \(11^4 = 11 \times 11 \times 11 \times 11\)
   Using the calculator to evaluate
   PRESS 11 \[x^4\] =
   \(11^4 = 14 641\)
   c \((-3)^2 = -3 \times -3 = 9\)

2 Using index law 1:
   a \(5^2 \times 5^4 = 5^2 + 4 = 5^6\)
   b \(a^2 \times a^4 = x^2 + 4 = x^6\)
   c \(3x^4 \times 5x^2y^2 = 3 \times 5 \times x^4 \times 2 \times y^1 = 15x^6y^3\)

3 Using index law 2:
   a \(a^3 + a = a^3 - 1 = a^4\)
   b \(16x^6 + 6x^3 = \frac{18x^6}{6x^3} = \frac{18}{6} \times \frac{x^6}{x^3} = 3x^3\)
   c \(24x^5y^3 \div 6xy = \frac{24x^5}{6y} \times \frac{x^5}{y} = 4x^4y^2\)

4 Using index law 3:
   a \((x^3)^3 = x^3 \times 3 = x^{13}\)
   b \((3x^3)^4 = 3^4 \times (x^3)^4 = 81 \times x^{12} = 81x^{12}\)
   c \((x^3)^4 + (x^2)^3 = x^{12} + x^6 = x^6\)

5 Using index law 4:
   a \(7^0 = 1\)
   b \(18x^3 + 6x^3 = \frac{18x^3}{6x^3} = \frac{3 \times x^3}{3x^3} = 3x^0 = 3 \times 1 = 3\)
   c \((2y^3)^4 + (4y^6)^2 = \frac{2^4 \times (y^3)^4}{4^2 \times (y^6)^2} = \frac{16 \times y^{12}}{16 \times y^{12}} = 1 \times y^{0} = 1\)

With practice, many of the steps in the above solutions can be left out.
Exercise 5:01

1. Simplify each expression by writing in index form.
   a. \(3^4\)             b. \(8^3\) 
   c. \(12^2\)            d. \(5^3\) 
   e. \(6^4\)            f. \((-3)^4\) 
   g. \(3^2 \times 2^3\)  h. \(4^3 \times 2^3\) 
   i. \(9^4 \times 3^4\)  j. \(25^2 \times 5^4\)

2. Determine the basic numeral for:
   a. \(5 \times 5 \times 5 \times 5\) 
   b. \(4 \times 4 \times 4\) 
   c. \(10 \times 10 \times 10 \times 10 \times 10 \times 10\) 
   d. \(x \times x \times x \times x \times x \times x\) 
   e. \(y \times y\)       f. \(x^3 \times x^2\) 
   g. \(a \times a^2 \times a^3\)  h. \((m^2)^3\) 
   i. \(y^7 \div y^2\)    j. \(\frac{w^3 \times w^3}{w^4}\)

3. Simplify the following, writing answers in index form.
   a. \(x^4 \times x^2\)  b. \(d^6 \times d\) 
   c. \(m^5 \times m^2\)  d. \(4a^6 \times 3a^3\) 
   e. \(-5y^2 \times 3y^3\) f. \(x^6 \times x^3\) 
   g. \(m^6 + m\)        h. \(a^6 \div a^3\) 
   i. \(16y^8 \div 4y^4\) j. \(27d^{12} \div (-3d^4)\) 
   k. \((x^3)^3\)        l. \((m^2)^3\) 
   m. \((3x^3)^2\)       n. \((4a)^4\) 
   o. \((6y^3)^3\)

4. Simplify:
   a. \(4^2 \times 4^3\)  b. \(5 \times 5^7\) 
   c. \(4 \times 2^5\)    d. \(10^4 \times 1000\) 
   e. \(3^9 \div 3^3\)   f. \(10^4 \div 10^4\) 
   g. \(3^3 \div 27\)    h. \((5^2)^3\) 
   j. \((2^3 \times 8)^2\) k. \((4 \times 16)^3\) 
   m. \((2^3 \times 8)^2\) n. \((42 \div 21)^6\)

5. Simplify:
   a. \(8x^4 \times x^3\)  b. \(5a^2 \times a\) 
   c. \(8x^4 + x^3\)    d. \(5a^2 + a\) 
   g. \(10y^3 \times 5y\)  h. \(16m^2 \times 2m^2\) 
   j. \(10y^3 \div 5y\)   k. \(16m^2 + 2m^2\) 
   m. \(12x^5 \times 6x^3\) n. \(9a^7 \div 3a^2\) 
   p. \(\frac{12x^3}{6x^3}\)  q. \(\frac{9a^7}{3a^2}\) 
   s. \(\frac{3a^5}{a^3}\)    t. \(\frac{10x^5}{5}\) 
   u. \(\frac{42a^7}{21a^6}\)
6. Simplify:
   \begin{align*}
   &a. \quad 6a^0 \\
   &b. \quad 6(a^3)^0 \\
   &c. \quad (6a^3)^0 \\
   &d. \quad ab^0 \\
   &e. \quad x^3y^3 \\
   &f. \quad m^3n^0 \\
   &g. \quad (2m^2)^3 \\
   &h. \quad (4n^4)^2 \\
   &i. \quad (2p^3)^4 \\
   &j. \quad x^2y^3 \times x^3 \\
   &k. \quad a^3b^3 \times b^4 \\
   &l. \quad xy^3 \times x^4 \\
   &m. \quad x^2y^2 \times xy^2 \\
   &n. \quad a^3b \times a^2b \\
   &o. \quad m^3n \times x^4 \\
   &p. \quad (x^2y^3)^2 \\
   &q. \quad (abc)^2 \\
   &r. \quad (pq)^3 \\
   &s. \quad 5x^2y \times 2xy \\
   &t. \quad 4a^2b^2 \times 7ab^3 \\
   &u. \quad 11a^3 \times 4a^2b^2 \\
   &v. \quad a \times a \times a \times 3a \\
   &w. \quad 3a \times 2a \times -4a \\
   &x. \quad 5c^4 \times 4c^2 + 10c^5 \\
   &y. \quad 12x^2 - 5x^3 + 10x^2 \\
   &z. \quad 4x(3x + 2) - (x - 1)
   \end{align*}

7. Simplify:
   \begin{align*}
   &a. \quad 5x^2 \times 2x^3 \times 3x \\
   &b. \quad 5 \times 2a \times 4a^2 \\
   &c. \quad 5y \times y^2 \times xy \\
   &d. \quad 4x^4 + 8x \\
   &e. \quad 7y^3 + 49y^2 \\
   &f. \quad 100x^3 + 10x^4 \\
   &g. \quad (x^2)^3 \times x^2 \\
   &h. \quad (a^3)^2 \times a^5 \\
   &i. \quad (y^7)^3 \times y^3 \\
   &j. \quad (a^3)^3 + a^4 \\
   &k. \quad (m^3)^4 + m^{10} \\
   &l. \quad n^8 + (n^2)^3 \\
   &m. \quad (y^3)^2 \times (y^3)^3 \\
   &n. \quad (2a^3)^3 \times (a^3)^2 \\
   &o. \quad (b^4)^3 \times (b^2)^3 \\
   &p. \quad (x^4 \times x^7) + x^9 \\
   &q. \quad (4a^3 \times 5a^4) + 10a^5 \\
   &r. \quad 7p^7q^5 + (p^2q)^3 \\
   &s. \quad 5x^3 \times 4x^7 \\
   &t. \quad \frac{(3x^3)^2 \times 4x^5}{6x^4 \times x} \\
   &u. \quad \frac{x^2 \times (xy)^3}{(2x)^4}
   \end{align*}

8. Expand and simplify:
   \begin{align*}
   &a. \quad x^2(x^2 - 1) \\
   &b. \quad a^3(5 - a^2) \\
   &c. \quad a^2(5a - a^3) \\
   &d. \quad x(x^2 + y) \\
   &e. \quad m(7 - m^4) \\
   &f. \quad y(y^2 - xy) \\
   &g. \quad 3a^2(2a^3 + 3a) \\
   &h. \quad 5x(3x^2 - x) \\
   &i. \quad 2m^3(n^2 - m^4) \\
   &j. \quad x(5x^2 - 3x + 7) \\
   &k. \quad x^2(2x^3 + 7x - 14) \\
   &l. \quad y(3y^2 + 7y - 1) \\
   &m. \quad x^2(x - 7) - x^3 \\
   &n. \quad y(4y^3 + 2) - 2y \\
   &o. \quad x(x^2 - 7x + 1) - (x^3 - x^2)
   \end{align*}

9. Simplify:
   \begin{align*}
   &a. \quad 3^x \times 3^x + 1 \\
   &b. \quad 5^{2y} + 5^y + 1 \\
   &c. \quad (2x)^2 + (2^{1 - x})^2 \\
   &d. \quad e^{2x + 1} \times e^x \\
   &e. \quad e^{2x + 1} + e^x \\
   &f. \quad (e^x + 2)^2 + e^x + 1
   \end{align*}
Consider the problem \( \frac{8}{32} \).

By dividing by the highest common factor, 8:

Also, \( 8^\frac{3}{5} = 2^3 + 2^5 \)

So \( \frac{8}{32} = \frac{2^3}{2^5} = 2^{3-5} = 2^{-2} \)

Therefore, \( 2^{-2} = \frac{1}{2^2} = \frac{1}{4} \).

In general, the meaning of a negative index can be summarised by the rules:

\( x^{-m} = \frac{1}{x^m}, \ (x \neq 0) \)

\( x^{-m} \) is the reciprocal of \( x^m \), since \( x^m \times x^{-m} = 1 \)

worked examples

1. Simplify the following, writing answers with only positive indices.
   - a) \( 5^{-2} \)
   - b) \( 7^{-1} \)
   - c) \( x^4 \times x^{-5} \)
   - d) \( 9a^3 + 3a^5 \)
   - e) \( 4y + 12y^3 \)
   - f) \( \left( \frac{2}{3} \right)^{-2} \)

2. Evaluate, using the calculator.
   - a) \( 3^{-2} \)
   - b) \( \left( \frac{3}{4} \right)^{-3} \)

Solutions

1. a) \( 5^{-2} = \frac{1}{5^2} = \frac{1}{25} \)
   - b) \( 7^{-1} = \frac{1}{7} \)
   - c) \( x^4 \times x^{-5} = x^{-1} \)
   - d) \( 9a^3 + 3a^5 = 3 \times a^2 \)
   - e) \( 4y + 12y^3 = \frac{4}{12}y^{1-3} = \frac{1}{3} \times y^{-2} \)
   - f) \( \left( \frac{3}{5} \right)^{-2} = \frac{1}{\left( \frac{3}{5} \right)^2} = \frac{1}{\left( \frac{9}{25} \right)} = \frac{25}{9} \)

2. a) \( 3^{-2} = \frac{1}{3^2} = \frac{1}{9} \)

On the calculator \( \boxed{3} \times 2 = 9 \)

continued ➔
Exercise 5:02

1. Write down the value of each of the following.
   a. $3^{-1}$  
   b. $5^{-1}$  
   c. $2^{-1}$  
   d. $6^{-2}$  
   e. $4^{-2}$  
   f. $10^{-3}$  
   g. $2^{-4}$  
   h. $10^{-4}$  
   i. $5^{-2}$

2. Write each with a negative index.
   a. $\frac{1}{11}$  
   b. $\frac{1}{3}$  
   c. $\frac{1}{5}$  
   d. $\frac{1}{7}$  
   e. $\frac{1}{33}$  
   f. $\frac{1}{54}$  
   g. $\frac{1}{28}$  
   h. $\frac{1}{72}$  
   i. $\frac{1}{10^2}$  
   j. $\frac{1}{10^3}$  
   k. $\frac{1}{10^6}$  
   l. $\frac{1}{10^5}$

3. Write true or false for:
   a. $1024 = 2^{10}$  
   b. $8 = 2^4$  
   c. $3^{-2} = \frac{1}{9}$  
   d. $2(3)^2 = 36$  
   e. $2(3)^{-1} = \frac{1}{6}$  
   f. $4^{-1} = \frac{1}{2}$  
   g. $2^{-1} < 1$  
   h. $-2^8 = (-2)^8$

4. Simplify, writing your answers without negative indices.
   a. $x^3 \times x^{-2}$  
   b. $a^{-2} \times a^5$  
   c. $m^4 \times m^{-1}$  
   d. $n^3 \times n^{-5}$  
   e. $3a^2 \times a^{-1}$  
   f. $6x^2 \times 5x^4$  
   g. $a^2 \times 5a^3$  
   h. $15m^{-1} \times 2m^3$  
   i. $x^{-4} \times x$  
   j. $2a^2 \times a^{-3}$  
   k. $4y \times 2y^{-2}$  
   l. $15m^{-4} \times 2m^{-1}$

5. Simplify, writing your answers without negative indices.
   a. $m^4 \div m^{-1}$  
   b. $x^2 \div x^{-2}$  
   c. $y^{-6} \div y^{-8}$  
   d. $x^{-3} \div x^{-1}$  
   e. $a^2 \div a^{-2}$  
   f. $y^{-1} \div y^3$  
   g. $y^{-2} \div y$  
   h. $x^{-3} \div x^{-1}$  
   i. $6x^2 \div 2x^{-1}$  
   j. $10a^2 + 5a^7$  
   k. $24a^{-2} + a^3$  
   l. $18n^{-1} + 9n^{-2}$

6. Simplify, writing your answers without negative indices.
   a. $(a^3)^{-2}$  
   b. $(x^2)^{-1}$  
   c. $(y^{-3})^2$  
   d. $(m^{-2})^{-2}$  
   e. $(2x^2)^{-1}$  
   f. $(3x)^{-2}$  
   g. $(5x^{-1})^2$  
   h. $(7x^{-2})^2$  
   i. $(abc)^{-1}$  
   j. $(a^3b^{-2}c)^{-1}$  
   k. $(2ab^{-1})^{-1}$  
   l. $(2a^2b)^{-1}$

7. If $x = 2$, $y = 3$ and $z = \frac{1}{2}$, evaluate:
   a. $x^{-1} + y^{-1}$  
   b. $(xy)^{-1}$  
   c. $(xz)^{-1}$  
   d. $x^{-1}y^{-1}z$

8. Simplify:
   a. $3x + 3^{-x}$  
   b. $5^y + 5^2 - y$  
   c. $e^x + 1 + e^{1-x}$  
   d. $(e^x)^2 \times e^{-(x-1)}$

* The formula for the volume of a sphere is: 
  $$V = \frac{4}{3} \pi r^3$$
  where $\pi \approx 3.142$ and $r$ is the radius of the sphere.
CHAPTER 5 INDICES AND SURDS

5:03 | Fractional Indices

What is the meaning of a fractional index?
The meaning is shown in the examples below.

1 \[ \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2} + \frac{1}{2}\right)^2 \]
   \[ = 9^1 \]
   \[ = 9 \]

That's neat!
\( (5^{\frac{1}{2}})^2 = 5 \)
That means that \( 5^{\frac{1}{2}} \) is the square root of 5.

2 \[ 5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\left(\frac{1}{2} + \frac{1}{2}\right)} \]
   \[ = 5^1 \]
   \[ = 5 \]

Now \( \sqrt{5} \times \sqrt{5} = 5 \)
So \( 5^{\frac{1}{2}} = \sqrt{5} \)

\[ \text{The number that multiplies itself to give } 5 \]
\[ \left(\text{ie } 5^{\frac{1}{2}}\right) \text{ is the square root of } 5. \]

3 Similarly:
\[ \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = 8^{\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right)} \]
   \[ = 8^1 \]
   \[ = 8 \]

So \( 8^{\frac{1}{3}} = \frac{1}{3}, \) (the cube root of 8)
\[ \therefore 8^{\frac{1}{3}} = 2 \]

Since \( (\frac{1}{x})^3 = x, \)
\[ \frac{1}{x} = x^3 \]
\[
x^2 = \sqrt{x}, \quad x^3 = \sqrt[3]{x}, \quad x^n = n\text{th root of } x
\]
\[
x^\frac{1}{n} \text{ is the number that, when used three times in a product, gives } x.
\]

**1** Simplify the following:

\begin{align*}
a. \quad 25^\frac{1}{2} & \quad b. \quad 27^\frac{1}{3} & \quad c. \quad 3x^3 \times 4x^2 \\
d. \quad (49m^6)^\frac{1}{2} & \quad e. \quad 8^\frac{3}{2} & \quad f. \quad 9^\frac{3}{2}
\end{align*}

**2** Evaluate using your calculator:

\begin{align*}
a. \quad 196^\frac{1}{2} & \quad b. \quad 32^\frac{1}{3} & \quad c. \quad 256^\frac{1}{4} & \quad d. \quad 125^\frac{1}{3}
\end{align*}

**Solutions**

\begin{align*}
a. \quad 25^\frac{1}{2} & = \sqrt{25} = 5 \\
b. \quad 27^\frac{1}{3} & = \sqrt[3]{27} = 3 \\
c. \quad 3x^3 \times 4x^2 & = 12x^5 \\
d. \quad (49m^6)^\frac{1}{2} & = \sqrt{49} \times m^3 = 7m^3 \\
e. \quad 8^\frac{3}{2} & = (8^3)^\frac{1}{2} = (\frac{1}{8})^\frac{1}{2} = 2^\frac{1}{2} = 2 \\
f. \quad 9^\frac{3}{2} & = (9^3)^\frac{1}{2} = (\frac{1}{9})^\frac{1}{2} = \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}
\end{align*}

Note from examples **e** and **f** the rule:

\[
\sqrt[\text{p}]{\text{x}^{\text{q}}} = \text{x}^{\frac{\text{q}}{\text{p}}} \quad \text{or} \quad (\sqrt[\text{p}]{\text{x}})^{\text{q}}
\]

**2** a. \(196^\frac{1}{2} = \sqrt{196} = 14\)

Using the square root key
\(\sqrt{196} = 14\)

b. To evaluate \(32^\frac{1}{5}\)

\[\text{Press: } 32 \quad \sqrt[5]{\text{y}} \quad 5 \quad = \]

Answer: \(32^\frac{1}{5} = 2\)

c. To evaluate \(256^\frac{1}{4}\)

\[\text{Press: } 256 \quad \sqrt[4]{\text{y}} \quad 4 \quad \sqrt[4]{\text{y}} \quad = \]

Answer: \(256^\frac{1}{4} = 0.25\)

d. To evaluate \(125^\frac{1}{3}\)

\[\text{Press 125} \quad \sqrt[3]{\text{y}} \quad 3 \quad \sqrt[3]{\text{y}} \quad 4 \quad \sqrt[3]{\text{y}} \quad 7 \quad \sqrt[3]{\text{y}} \quad = \]

Answer: \(125^\frac{1}{3} = 0.0016\)

**Exercise 5:03**

1. Write each of the following using a square root sign.

\begin{align*}
a. \quad 5^\frac{1}{2} & \quad b. \quad 10^\frac{1}{2} & \quad c. \quad 2^\frac{1}{2} \\
d. \quad 3 \times 2^\frac{1}{2} & \quad e. \quad 4 \times 3^\frac{1}{2} & \quad f. \quad 7 \times 6^\frac{1}{2}
\end{align*}

2. Use a fractional index to write:

\begin{align*}
a. \quad \sqrt{3} & \quad b. \quad 3\sqrt{2} & \quad c. \quad \sqrt[3]{11} \\
d. \quad 7\sqrt[3]{3}
\end{align*}
3. Find the value of the following:
   \[ a = 4^{\frac{1}{2}} \quad b = \frac{40}{3} \quad c = 8^{\frac{1}{3}} \quad d = 16^{\frac{1}{4}} \]
   \[ e = 16^{\frac{1}{2}} \quad f = 100^{\frac{1}{2}} \quad g = 144^{\frac{1}{2}} \quad h = 1^{\frac{1}{2}} \]
   \[ i = 121^{\frac{1}{2}} \quad j = 32^{\frac{3}{5}} \quad k = 81^{\frac{1}{2}} \quad l = 81^{\frac{1}{4}} \]

4. Assuming that all pronumerals used are positive, simplify:
   \[ a = x^{\frac{2}{3}} \times x^{\frac{1}{2}} \quad b = a^{\frac{1}{3}} \times a^{\frac{2}{3}} \quad c = m^{\frac{1}{2}} \times m^{\frac{1}{3}} \]
   \[ d = 6x^{\frac{1}{2}} \times 2x^{\frac{1}{2}} \quad e = 3y^{\frac{1}{4}} \times 2y^{\frac{3}{4}} \quad f = 9n^{\frac{2}{3}} \times 2n^{\frac{1}{3}} \]
   \[ g = (x^{\frac{1}{2}})^{\frac{1}{2}} \quad h = (y^{\frac{1}{3}})^{\frac{1}{3}} \quad i = (4a^{\frac{1}{2}})^{\frac{1}{2}} \]
   \[ j = (a^{2}b^{4})^{\frac{1}{2}} \quad k = (9x^{4}y^{6})^{\frac{1}{2}} \quad l = (8x^{2}y^{3})^{\frac{1}{2}} \]

5. Evaluate:
   \[ a = 9^{\frac{1}{2}} \quad b = 25^{\frac{1}{2}} \quad c = 8^{\frac{1}{3}} \]
   \[ d = 9^{\frac{1}{2}} \quad e = 4^{\frac{2}{3}} \quad f = 4^{\frac{3}{2}} \]
   \[ g = 16^{\frac{1}{2}} \quad h = 125^{\frac{2}{3}} \quad i = 8^{\frac{2}{3}} \]
   \[ j = 9^{\frac{1}{2}} \quad k = 32^{\frac{4}{5}} \quad l = 16^{\frac{3}{4}} \]

6. Evaluate using your calculator, leaving answers as decimal numerals:
   \[ a = 225^{\frac{1}{2}} \quad b = 784^{\frac{1}{3}} \quad c = 1024^{\frac{1}{3}} \]
   \[ d = 729^{\frac{1}{3}} \quad e = 3375^{\frac{1}{3}} \quad f = 8000^{\frac{1}{3}} \]
   \[ g = 225^{\frac{2}{3}} \quad h = 729^{\frac{1}{3}} \quad i = 8000^{\frac{1}{3}} \]
   \[ j = (0.125)^{\frac{2}{3}} \quad k = (0.25)^{\frac{5}{2}} \quad l = (0.01)^{\frac{3}{2}} \]

7. If \( a = 4 \), \( b = 8 \) and \( c = 9 \), evaluate the following:
   \[ a = a^{\frac{1}{2}} + b^{\frac{1}{3}} \quad b = (ab)^{\frac{1}{3}} \quad c = 2c^{\frac{1}{2}} \]
   \[ e = a^{\frac{1}{2}} + b^{\frac{1}{3}} \quad f = (2b)^{\frac{1}{4}} \quad g = (2ab)^{\frac{1}{6}} \]
   \[ d = \frac{1}{2}(ac)^{\frac{1}{2}} \quad h = (2bc)^{\frac{1}{2}} - a^{\frac{1}{2}} \]

8. Use the fact that \( x^{\frac{1}{2}} = \sqrt{x} \), to simplify:
   \[ a = (27a^{3})^{\frac{1}{3}} \quad b = (x^{6}y^{12})^{\frac{1}{2}} \quad c = (8m^{9})^{\frac{1}{2}} \]
   \[ d = \left(\frac{a^{3}}{b^{3}}\right)^{\frac{1}{2}} \quad e = \left(\frac{16}{x^{4}}\right)^{\frac{3}{2}} \quad f = \left(\frac{y^{6}}{25}\right)^{\frac{1}{2}} \]

As \( x^{\frac{1}{2}} = \sqrt{x} \), \( x^{\frac{1}{2}} \) stands for the positive square root of \( x \).
Note \( \sqrt{x^{2}} = |x| \).
Fun Spot 5:03 | Why is a room full of married people always empty?

Work out the answer to each part and put the letter for that part in the box that is above the correct answer.

Write in index form:

- Find the value of:  
  - E $3^2$  
  - A $10^3$

- Find the value of $x$ in:
  - I $2^x = 16$  
  - I $3^x = 9$

- Write as a basic numeral:
  - I $1.7 \times 10^2$  
  - I $6.04 \div 10$

Evaluate:

- O $x^0$  
- N $2^{-\frac{1}{2}}$  
- B $27^3$

- H To fill a jar in 6 minutes, Jan doubled the number of peanuts in the jar every minute. After how many minutes was the jar half full?

Simplify:

- S $3x - x$  
- T $x^3 \times x^3$  
- S $\frac{4aab}{ab}$

Investigation 5:03 | Reasoning with fractional indices

Please use the Assessment Grid on the following page to help you understand what is required for this Investigation.

- Write $\sqrt[3]{x}$, $\sqrt[4]{x^2}$, $\sqrt[6]{x^3}$, $\sqrt{x^5}$, $\sqrt[3]{x^7}$, ... as expressions with fractional indices and describe the pattern that emerges.
- Find the value of $b$ if $(x^b)^3 = x$
- Explain why $\sqrt[3]{8} = 2^{\frac{3}{2}} = 2 \sqrt{2} = (\sqrt{2})^3$.
- Find some values that $x$, $p$ and $q$ could take if $x^{\frac{p}{q}} = 2$. 

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

<table>
<thead>
<tr>
<th>Assessment Criteria (B, C) for this investigation</th>
<th>Achieved</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Criterion B Investigating Patterns</strong></td>
<td></td>
</tr>
<tr>
<td>a  None of the following descriptors has been achieved.</td>
<td>0</td>
</tr>
<tr>
<td>b  Some help was needed to be able to write the fractional indices.</td>
<td>1</td>
</tr>
<tr>
<td>c  Mathematical techniques have been selected and applied to write each fractional index and suggest an emerging pattern.</td>
<td>3</td>
</tr>
<tr>
<td>d  The student has completed all fractional indices and accurately described the rules for the square of a binomial. Some attempt at the final two parts has been made.</td>
<td>5</td>
</tr>
<tr>
<td>e  The above has been completed with specific justification using the patterns and index lawns shown and the further questions have been completed accurately.</td>
<td>7</td>
</tr>
<tr>
<td><strong>Criterion C Communication in Mathematics</strong></td>
<td></td>
</tr>
<tr>
<td>a  None of the following descriptors has been achieved.</td>
<td>0</td>
</tr>
<tr>
<td>b  There is a basic use of mathematical language and notation, with some errors or inconsistencies evident. Lines of reasoning are insufficient.</td>
<td>1</td>
</tr>
<tr>
<td>c  There is sufficient use of mathematical language and notation. Explanations are clear but not always complete.</td>
<td>3</td>
</tr>
<tr>
<td>d  Correct use of mathematical language and notation has been shown. Explanations of all rules are complete and concise.</td>
<td>5</td>
</tr>
</tbody>
</table>
Scientific (or Standard) Notation

Investigation 5.04 | Multiplying and dividing by powers of 10

Please use the Assessment Grid on the following page to help you understand what is required for this Investigation.

- Use the x^y button on your calculator to answer these questions.
- Look for a connection between questions and answers and then fill in the rules at the end of the investigation.

Exercise

1.
- a | $1.8 \times 10^1$
- b | $1.8 \times 10^2$
- c | $1.8 \times 10^3$
- d | $4.05 \times 10^1$
- e | $4.05 \times 10^2$
- f | $4.05 \times 10^3$
- g | $6.2 \times 10^4$
- h | $6.2 \times 10^5$
- i | $6.2 \times 10^6$
- j | $3.1416 \times 10^2$
- k | $3.1416 \times 10^3$
- l | $3.1416 \times 10^4$

To multiply by $10^n$ move the decimal point _____ places to the _____.

2.
- a | $1.8 \div 10^1$
- b | $1.8 \div 10^2$
- c | $1.8 \div 10^3$
- d | $968.5 \div 10^2$
- e | $968.5 \div 10^3$
- f | $968.5 \div 10^4$

To divide by $10^n$ move the decimal point _____ places to the _____.

The investigation above should have reminded you that:
1. when we multiply a decimal by 10, 100 or 1000, we move the decimal point 1, 2 or 3 places to the right.
2. when we divide a decimal by 10, 100 or 1000, we move the decimal point 1, 2 or 3 places to the left.

When expressing numbers in scientific (or standard) notation each number is written as the product of a number between 1 and 10, and a power of 10.

- This number is written in scientific notation (or standard form).
- The first part is between 1 and 10.
- The second part is a power of 10.

Scientific notation is useful when writing very large or very small numbers.

Numbers greater than 1

$5.97 \times 10^3 = 5970$

To write 5970 in standard form:
- put a decimal point after the first digit
- count the number of places you have to move the decimal point to the left from its original position.
This will be the power needed.
Assessment Grid for Investigation 5:04 | Multiplying and dividing by powers of 10

The following is a sample assessment grid for this investigation. You should carefully read the criteria before beginning the investigation so that you know what is required.

<table>
<thead>
<tr>
<th>Assessment Criteria (B, C) for this investigation</th>
<th>Achieved</th>
</tr>
</thead>
<tbody>
<tr>
<td>Criterion B Investigating Patterns</td>
<td></td>
</tr>
<tr>
<td>a None of the following descriptors has been achieved.</td>
<td>0</td>
</tr>
<tr>
<td>b Some help was needed to complete the exercises.</td>
<td>1</td>
</tr>
<tr>
<td>c The student independently completes the exercises.</td>
<td>3</td>
</tr>
<tr>
<td>d The student has correctly done all exercises and attempted to complete the rules following.</td>
<td>5</td>
</tr>
<tr>
<td>e All exercises and rules are completed correctly with thorough explanation and justification, possibly with further examples for support.</td>
<td>7</td>
</tr>
<tr>
<td>Criterion C Communication in Mathematics</td>
<td></td>
</tr>
<tr>
<td>a None of the following descriptors has been achieved.</td>
<td>0</td>
</tr>
<tr>
<td>b There is a basic use of mathematical language and notation, with some errors or inconsistencies evident. Lines of reasoning are insufficient.</td>
<td>1</td>
</tr>
<tr>
<td>c There is sufficient use of mathematical language and notation. Explanations are clear and the student generally moves between the different forms of representation well.</td>
<td>3</td>
</tr>
<tr>
<td>d Correct use of mathematical language and notation has been shown with complete working. Explanations of all rules are complete and the student easily moves between the different forms of representation.</td>
<td>5</td>
</tr>
</tbody>
</table>
1 Express the following in scientific notation.
   a 243 
   b 60 000 
   c 93 800 000

2 Write the following as a basic numeral.
   a $1.3 \times 10^2$
   b $2.431 \times 10^2$
   c $4.63 \times 10^7$

Solutions

1 a $243 = 2.43 \times 100$ 
   = $2.43 \times 10^2$
   
   b $60 000 = 6 \times 10 000$
   = $6 \times 10^4$
   
   c $93 800 000 = 9.38 \times 10 000 000$
   = $9.38 \times 10^7$

2 a $1.3 \times 10^2 = 1.30 \times 100$
   = 130
   
   b $2.431 \times 10^2 = 2.431 \times 100$
   = 243.1
   
   c $4.63 \times 10^7 = 4.630 000 0 \times 10 000 000$
   = 46 300 000

If end zeros are significant, write them in your answer.
Eg 60 000 (to nearest 100) = $6.00 \times 10^4$

Numbers less than 1

$0.005 97 = 5.97 \times 10^{-3}$

To write 0.005 97 in scientific notation:

- put a decimal point after the first non-zero digit
- count the number of places you have moved the decimal point to the right from its original position.
  This will show the negative number needed as the power of 10.

$5.97 \times 10^{-3}$ is the same as $5.97 \div 10^3$.

worked examples

1 Express each number in scientific notation.
   a 0.043 
   b 0.000 059 7 
   c 0.004

2 Write the basic numeral for:
   a $2.9 \times 10^{-2}$
   b $9.38 \times 10^{-3}$
   c $1.004 \times 10^{-3}$

• How many places must we move the decimal point for scientific notation?
  Answer = 2

• Is 0.043 bigger or smaller than 1?
  Answer = smaller

• So the power of 10 is ‘−2’.
  $\therefore 0.043 = 4.3 \times 10^{-2}$
Solutions

1. a. \(0.043 = 4.3 \times 10^{-2}\)
   
   b. \(0.000\,059\,7 = 5.97 \times 10^{-5}\)

   c. \(0.004 = 4 \times 10^{-3}\)

2. a. \(2.9 \times 10^{-2} = 0.029 \times 100\)

   b. \(9.38 \times 10^{-5} = 0.000\,093\,8 \times 100\,000\)

   c. \(1.004 \times 10^{-3} = 0.001\,004 + 1000\)

   = 0.001\,004

Exercise 5:04

1. a. Explain the difference between \(2 \times 10^4\) and \(2^4\).

   b. Explain the difference between \(5 \times 10^{-2}\) and \(5^{-2}\).

   c. How many seconds are in 50,000 years?

   d. Have you lived \(8.2 \times 10^4\) hours?

   e. Order the following, from smallest to largest.

      \[
      \begin{align*}
      3.24 \times 10^3 & \quad 6 & \quad 9.8 \times 10^{-5} & \quad 5.6 \times 10^{-2} \\
      1.2 \times 10^4 & \quad 2.04 & \quad 5.499 \times 10^2 & \quad 0.0034
      \end{align*}
      \]

   f. Write the thickness of a sheet of paper in scientific notation if 500 sheets of paper have a thickness of 3.8 cm.

   g. Estimate the thickness of the cover of this book. Write your estimate in scientific notation.

2. Write the basic numeral for:

   a. \(2.1 \times 10^1\)

   b. \(2.1 \div 10^1\)

   c. \(2.1 \times 10^{-1}\)

   d. \(7.04 \times 10^2\)

   e. \(7.04 \div 10^2\)

   f. \(7.04 \times 10^{-2}\)

   g. \(1.375 \times 10^3\)

   h. \(1.375 + 10^3\)

   i. \(1.375 \times 10^{-3}\)

3. Express in scientific notation.

   (Assume that final zeros are not significant.)

   a. 470

   b. 2600

   c. 53000

   d. 700

   e. 50000

   f. 700000

   g. 65

   h. 342

   i. 90

   j. 4970

   k. 63500

   l. 2941000

   m. 297.1

   n. 69.3

   o. 4976.5

   p. 9310000

   q. 67000000

   r. 190100

   s. 600000

   t. 501700

   u. 100000

4. Express in scientific notation.

   a. 0.075

   b. 0.0063

   c. 0.59

   d. 0.08

   e. 0.0003

   f. 0.009

   g. 0.3

   h. 0.0301

   i. 0.000529

   j. 0.426

   k. 0.001

   l. 0.0000097

   m. 0.00006

   n. 0.000907

   o. 0.000000004

If you’re stuck with this exercise, think back to Investigation 6:02...
Write the basic numeral for:

a \[2 \times 3 \times 10^2\]

b \[9.4 \times 10^3\]

c \[3.7 \times 10^3\]

d \[2.95 \times 10^2\]

e \[8.74 \times 10^1\]

f \[7.63 \times 10^5\]

g \[1.075 \times 10^3\]

h \[2.0 \times 10^4\]

i \[8 \times 10^4\]

j \[2.9 \times 10^-2\]

k \[1.9 \times 10^-3\]

l \[9.5 \times 10^-1\]

m \[3.76 \times 10^-3\]

n \[4.63 \times 10^-4\]

o \[1.07 \times 10^-2\]

p \[7 \times 10^-2\]

q \[8.0 \times 10^-1\]

r \[5 \times 10^-6\]

s \[9.73 \times 10^3\]

t \[6.3 \times 10^-3\]

u \[4.7 \times 10^7\]

v \[9.142 \times 10^2\]

w \[1.032 \times 10^-2\]

x \[1.0 \times 10^8\]

5:05 | Scientific Notation and the Calculator

Write in scientific notation:

1. 690
2. 4000
3. 963.2
4. 0.073
5. 0.0003

Rewrite as basic numerals:

6. 2.9 \times 10^3
7. 8.0 \times 10^5
8. 4.6 \times 10^-2
9. 5 \times 10^-7
10. 8.14 \times 10^{-1}

On a calculator:

5.517 \times 10^{12} is shown as 
3.841 \times 10^{-6} is shown as

This is the calculator’s way of showing scientific notation.

To enter scientific notation, press:

5.517 \& Exp \quad 12, \text{ to enter } 5.517 \times 10^{12}, \text{ and }
3.841 \& Exp \quad 6 \quad \pm, \text{ to enter } 3.841 \times 10^{-6}.

To convert calculator answers into decimal form:

1. Locate the decimal point in the first part of the number (the part between 1 and 10).
2. Look at the sign of the second part. This tells you in which direction to move the decimal point. If it is negative the point moves to the left. If it is positive the point moves to the right.
3. Look at the size of the second part. This tells you how many places the decimal point has to be moved.
4. Move the decimal point to its new position, filling in any gaps, where necessary, with zeros.

Some calculators are called ‘Scientific Calculators’ because they can give answers in scientific (or standard) notation.

Use a calculator to find the answers for:

1. \[630000 \times (47000)^2\]
2. \[45 \div (8614)^3\]
3. \[(8.4 \times 10^6) + (3.8 \times 10^7)\]
4. \[\sqrt{1.44 \times 10^{-6}}\]

A calculator will give an answer in scientific notation if the number is too large or small to fit on the screen.
Solutions

1 \[ 630000 \times (47000)^2 \]
\[= 1.39167 \times 10^{15} \]
\[= 1.39167 \times 10^{15} \]
\[= 1391670000000000 \]

2 \[ 45 + (8614)^3 \]
\[= 7.040409359 \times 10^{-11} \]
\[= 7.040409359 \times 10^{-11} \]
\[= 0.00000000070409359 \]

3 \[ (8.4 \times 10^6) + (3.8 \times 10^7) \]
Press: \[8 \times \text{Exp} 6 + 3.8 \times \text{Exp} 7 \]
\[= 4640000 \]

4 \[ \sqrt{1.44 \times 10^{-6}} \]
Press: \[\text{Exp} 6 \div \text{Exp} 7 \]
\[= 0.0012 \]

Note: Not all calculators work the same way.

Exercise 5:05

1 Enter each of these on your calculator using the \[\text{Exp}\] key, and copy the calculator readout.
   a \[ 6.3 \times 10^{15} \]
   b \[ 1.4 \times 10^{-12} \]
   c \[ 9.2 \times 10^{11} \]

2 Rewrite these calculator readouts in scientific notation using powers of 10.
   a \[ 3.02 \times 10^{-5} \]
   b \[ 4.631 \times 10^{-6} \]
   c \[ 1.371 \times 10^{-3} \]
   d \[ 1.31 \times 10^{-4} \]
   e \[ 6.9 \times 10^{-8} \]
   f \[ 4.6327 \times 10^{-10} \]
   g \[ 7.6514 \times 10^{-8} \]
   h \[ 1.03124 \times 10^{-9} \]
   i \[ 6.9333 \times 10^{-9} \]

   Explain why a calculator readout of \[2.04 \times 10^{-5} \]
   has a different value to \[2^4 \].

3 Place the nine numbers in question 2 in order of size from smallest to largest.

4 Give the answers to these in scientific notation, correct to 5 significant figures.
   a \[ 3814^4 \]
   b \[ 0.0004 \div 8400^2 \]
   c \[ (0.007)^5 \]
   d \[ 93000000 \div 0.00013 \]
   e \[ (65 \times 847)^3 \]
   f \[ (0.0045)^3 \div (0.0038)^2 \]
   g \[ 9865 \times 8380 \]
   h \[ 6800 \div (0.0007)^5 \]

5 Use a calculator to answer correct to 4 significant figures, then use the index laws to check your answer.
   a \[ 13.83 \times (2.3 \times 10^4) \]
   b \[ (8.14 \times 10^{-2})^2 \]
   c \[ (2.1 \times 10^6) + (8.6 \times 10^8) \]
   d \[ (3.8 \times 10^{-3})^2 \]
   e \[ 468 \times (1.8 \times 10^{-3}) \]
   f \[ (9.1 \times 10^4) + (6.8 \times 10^5) \]
   g \[ \sqrt[3]{7.45 \times 10^9} \]
   h \[ \frac{3}{9.1} \times 10^{-8} \]
   i \[ \frac{3}{6.714} \times 10^{-12} \]

6 a An American reported that the diameter of the sun is approximately \[8.656 \times 10^5 \] miles.
   Write this in kilometres, using scientific notation written correct to four significant figures.
   There are \[1.609 \text{ km in a mile} \]. If the sun’s diameter is \[109 \text{ times that of the earth} \], what is the earth’s diameter, correct to three significant figures?

   b The distance to the sun varies from \[1.47 \times 10^8 \] km in January to \[1.52 \times 10^8 \] km in July. This is because the earth’s orbit is an ellipse. What is the difference between these distances?
If we use the average distance to the sun \((1.50 \times 10^8 \text{ km})\), how long would it take light travelling at \(3.0 \times 10^8 \text{ m/s}\) to reach the earth? (Answer correct to the nearest minute.)

The mass of the earth is approximately \(6 \times 10^{21} \text{ tonnes}\). The sun's mass is about 333,400 times greater than the mass of the earth. What is the mass of the sun correct to one significant figure?

We belong to the galaxy known as the Milky Way. It contains about \(1 \times 10^{11}\) stars. If the sun is taken to have average mass [see part d], what is the total mass, correct to 1 significant figure, of the stars in the Milky Way?

Investigation 5:05 | Using scientific notation

1. The speed of light is \(3.0 \times 10^8 \text{ m/s}\). Use reference books and your calculator to complete this table for five stars of your choice (e.g., Vega, Dog Star, Pole Star, Sirius).

<table>
<thead>
<tr>
<th>Name of star</th>
<th>Distance from the earth</th>
<th>Time taken for light to travel to the earth</th>
</tr>
</thead>
<tbody>
<tr>
<td>The sun</td>
<td>(1.5 \times 10^8 \text{ km})</td>
<td></td>
</tr>
<tr>
<td>Alpha Centauri</td>
<td>(4.2 \times 10^{13} \text{ km})</td>
<td></td>
</tr>
</tbody>
</table>

Order the distances of the five stars from the earth, from smallest to largest.

2. Research nanotechnology, which involves the use of very small machine parts. Parts are often measured in micrometres. Make comparisons between the sizes of components.

- Distances in astronomy are measured in light years, which is the distance that light travels in a year. A light year is approximately \(9.6 \times 10^{12} \text{ km}\). 

10^6 is 1 million, 10^9 is 1 billion.
The Real Number System

The real number system is made up of two groups of numbers: rational and irrational numbers.

**Rational numbers**

Any number that can be written as a fraction, \( \frac{a}{b} \) where \( a \) and \( b \) are whole numbers and \( b \neq 0 \), is a rational number. These include integers, fractions, mixed numbers, terminating decimals and recurring decimals.

eg \( \frac{7}{8}, 6\frac{3}{5}, 1.25, 0.07, 0.\dot{4}, \sqrt{81} \)

These examples can all be written as fractions.

\( \frac{7}{8}, \frac{33}{5}, \frac{7}{4}, \frac{9}{100}, \frac{6}{1} \)

*Note: An integer is a rational number whose denominator is 1.*

**Irrational numbers**

It follows that irrational numbers cannot be written as a fraction, \( \frac{a}{b} \) where \( a \) and \( b \) are whole numbers. We have met a few numbers like this in our study of the circle and Pythagoras’ theorem.

eg \( \pi, \sqrt{2}, \sqrt[3]{4}, \sqrt{3} + 2 \)

The calculator can only give approximations for these numbers. The decimals continue without terminating or repeating.

\( 3.14159265 \ldots, 1.41421356 \ldots, 1.58740105 \ldots, 3.73205080 \ldots \)

**Irrational numbers on the number line**

Although irrational numbers cannot be given an exact decimal value, their positions can still be plotted on the number line. As can be seen from exercises on Pythagoras’ theorem, a number such as \( \sqrt{2} \) can correspond to the length of a side of a triangle, and so this length can be shown on a number line.

Examine the diagram on the right.

If the length of the hypotenuse of this triangle is transferred, using compasses, to the number line as shown, we have the position of \( \sqrt{2} \) on the number line. (This agrees with the decimal approximation from the calculator of 1.414213 56.)

The previous construction can be extended to give the position of other square roots on the number line.

Another irrational number you have met before is \( \pi \). You should know that it has an approximate value of 3.142, so it would lie on the number line in the position shown.
Exercise 5:06

1 For each number write rational or irrational. (A calculator might help you to decide.)

\[
\begin{align*}
a & = \frac{7}{10} & b & = \sqrt{2} & c & = 0.3 & d & = \frac{11}{2} \\
e & = 1.6 & f & = \pi & g & = \sqrt{5} & h & = \frac{3}{4} \\
i & = \sqrt{16} & j & = 0.99 & k & = \frac{22}{7} & l & = 9 \\
m & = \sqrt{625} & n & = 0.666 & o & = 70 & p & = \sqrt{13} \\
q & = 1\frac{1}{3} & r & = 1 + \sqrt{3} & s & = \sqrt{1} & t & = \sqrt{4} + \sqrt{5} \\
u & = 0.0005 & v & = 1 - \sqrt{5} & w & = -\sqrt{3} & x & = \sqrt{7} + \sqrt{2} \\
y & = \frac{3}{27} & z & = \frac{3}{10}
\end{align*}
\]

2 Use your calculator to find an approximation correct to 1 decimal place for the following. Also use these values to show the position of each number on the number line.

\[
\begin{align*}
a & = \sqrt{2} & b & = \sqrt{3} & c & = \sqrt{5} & d & = \sqrt{6} & e & = \sqrt{7} \\
f & = \sqrt{8} & g & = \sqrt{10} & h & = \sqrt{12} & i & = \sqrt{20} & j & = \pi
\end{align*}
\]

3 Between which two consecutive integers does each number below lie?

\[
\begin{align*}
a & = \sqrt{11} & b & = \sqrt{18} & c & = \sqrt{41} & d & = \sqrt{78} & e & = \sqrt{95} \\
f & = \sqrt{125} & g & = \sqrt{180} & h & = \sqrt{250} & i & = \sqrt{390} & j & = \sqrt{901}
\end{align*}
\]

4 Arrange each set of numbers below in order, from smallest to largest.

\[
\begin{align*}
a & = \sqrt{5}, 2, \sqrt{3} & b & = \sqrt{8}, 3, \pi & c & = \sqrt{10}, \sqrt{12}, 3 \\
d & = 7, \sqrt{40}, \sqrt{50}, 6.5 & e & = \pi, \sqrt{7}, 2.1, \sqrt{12} & f & = 5.6, \sqrt{26}, 6, \sqrt{30} \\
g & = 8.1, \sqrt{65}, 7.9, \sqrt{60} & h & = \sqrt{98}, 10, \sqrt{102}, 10.1 & i & = 3.1, \pi, \sqrt{9}, 3.2 \\
j & = \sqrt{20}, 4.1, 4.5, \sqrt{21} & k & = 20, \sqrt{390}, 21, \sqrt{420} & l & = \sqrt{600}, \sqrt{610}, 24, 25
\end{align*}
\]

5 This diagram shows another construction for locating square roots on the number line.

Use a set square to draw the triangles on graph paper, then use compasses to draw the arcs on the number line.

Extend your diagram to show \( \sqrt{6} \).

Check the accuracy of your constructions with your calculator.
To show multiples of a square root on the number line we can use a pair of compasses to mark off equal intervals.

a Repeat the instructions in question 5 to find the position of $\sqrt{2}$ on the number line. Then use a pair of compasses to mark the position of $2\sqrt{2}$ and $3\sqrt{2}$.

b Draw a diagram and show the position of $\sqrt{3}$, $2\sqrt{3}$ and $-\sqrt{3}$ on a number line.

The position of $\pi$ on the number line can be shown by doing the following. Use the diameter of a ten cent coin to mark off units on a number line.

Then mark a point on the circumference of the coin, align it with zero on the number line, then roll it along the line carefully until the mark meets the number line again. This will show the position of $\pi$ on the number line.

---

**Proof that $\sqrt{2}$ is irrational**

Let us suppose that $\sqrt{2}$ is rational and, therefore, can be written as a fraction in the form $\frac{p}{q}$ where $p$ and $q$ are positive integers with no common factor. (This assumption is essential.)

so $\sqrt{2} = \frac{p}{q}$

then $2 = \frac{p^2}{q^2}$ (squaring both sides)

and $2q^2 = p^2$
This last step implies that $p^2$ must be divisible by 2 (since 2 is prime). Therefore 2 must divide into $p$ exactly.

$\therefore p$ can be expressed in the form $2k$ for some integer $k$.

$\therefore 2q^2 = (2k)^2$
$2q^2 = 4k^2$
$q^2 = 2k^2$

Now, as for $p$ above, it can be argued from this last step that $q$ must be divisible by 2. But $p$ and $q$ were said to have no common factor, hence a contradiction exists. So our original assumption was wrong.

Therefore $p$ and $q$ cannot be found so that $\sqrt{2} = \frac{p}{q}$. Hence $\sqrt{2}$ must be irrational.

- Try to use the method above to prove that these are irrational.

$\sqrt{3} \quad 2 \quad \sqrt{5} \quad 3 \quad \sqrt{11}$

**Challenge 5:06 | f-stops and $\sqrt{2}$**

Professional photographers have cameras that can alter shutter time and aperture settings using what are called f-stops.

The f-stops 2, 4, 8 and 16 are accurate. More accurate readings for the rest are given below the scale.

- Find the pattern in the accurate f-stops.
  * Try squaring each accurate f-stop number.
  * Try dividing each f-stop number by the one before it.
- Try to discover how f-stops are used.
CHAPTER 5 INDICES AND SURDS

5:07 | Surds

Find the value of:

1 \(\sqrt{16}\)  
2 \(\sqrt{9}\)  
3 \(\sqrt{36}\)  
4 \(\sqrt{16+9}\)  
5 \(\sqrt{16+\sqrt{9}}\)  
6 \(\sqrt{16} \times \sqrt{9}\)  
7 \(\sqrt{16} \times \sqrt{9}\)  
8 \(\sqrt{\frac{36}{9}}\)  
9 \(\sqrt{\frac{36}{\sqrt{9}}}\)  
10 \((\sqrt{16})^2\)

Surds are numerical expressions that involve irrational roots. They are irrational numbers.

Surds obey the following rules, which are suggested by Prep Quiz 5:07.

**Rule 1** \(\sqrt{xy} = \sqrt{x} \times \sqrt{y}\)

**worked examples**

1 \(\sqrt{100} = \sqrt{4} \times \sqrt{25}\)  
   = \(2 \times 5\)  
   = 10 (which is true)

2 \(\sqrt{27} = \sqrt{9} \times \sqrt{3}\)  
   = \(3 \times \sqrt{3}\)

3 \(\sqrt{5} \times \sqrt{7} = \sqrt{35}\)

**Note:** \(\sqrt{x}\) means the positive square root of \(x\) when \(x > 0\).
\(\sqrt{x} = 0\) when \(x = 0\).

**Rule 2** \(\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}\)

**worked examples**

1 \(\frac{\sqrt{16}}{\sqrt{4}} = \sqrt{\frac{16}{4}}\)  
   \(\text{ie} \quad \sqrt{4} = \frac{4}{2}\)  
   = \(2\) (which is true)

2 \(\sqrt{125} \div \sqrt{5} = \sqrt{\frac{125}{5}}\)  
   = \(\sqrt{25}\)  
   = 5

3 \(\sqrt{30} \div \sqrt{5} = \sqrt{\frac{30}{5}}\)  
   = \(\sqrt{6}\)

**Rule 3** \((\sqrt{x})^2 = x\)

**Note:** For \(\sqrt{x}\) to exist, \(x\) cannot be negative.

**worked examples**

1 \((\sqrt{25})^2 = (5)^2\)  
   = 25

2 \((\sqrt{7})^2 = 7\)

3 \((3\sqrt{2})^2 = 3^2 \times (\sqrt{2})^2\)  
   = 9 \times 2  
   = 18
A surd is in its simplest form when the number under the square root sign is as small as possible. To simplify a surd we make use of Rule 1 by expressing the square root as the product of two smaller square roots, one being the root of a square number. Examine the examples below.

**worked examples**

Simplify the following surds.

1. \( \sqrt{18} = \sqrt{9} \times \sqrt{2} \)
   
2. \( \sqrt{75} = \sqrt{25} \times \sqrt{3} \)

3. \( 5 \sqrt{48} = 5 \times \sqrt{16} \times \sqrt{3} \)

\[ 
1 = 3 \times \sqrt{2} \\
2 = 5 \times \sqrt{3} \\
3 = 20 \sqrt{3} 
\]

**Exercise 5:07**

1. Simplify:
   
   \[
   a \sqrt{3} \times \sqrt{5} \\
b \sqrt{5} \times \sqrt{3} \\
c \sqrt{7} \times \sqrt{6} \\
d \sqrt{6} \times \sqrt{7} \\
e \sqrt{10} \times \sqrt{3} \\
f \sqrt{23} \times \sqrt{2} \\
g \sqrt{13} \times \sqrt{5} \\
h \sqrt{11} \times \sqrt{3} \\
i \sqrt{5} \times \sqrt{2} \\
j \sqrt{7} \times \sqrt{2} \\
k \sqrt{11} \times \sqrt{10} \\
l \sqrt{13} \times \sqrt{7} \\
m \sqrt{26} + \sqrt{2} \\
n \sqrt{55} + \sqrt{5} \\
o \sqrt{77} + \sqrt{11} \\
p \sqrt{34} + \sqrt{17} \\
q \frac{\sqrt{3}}{\sqrt{2}} \\
r \frac{\sqrt{57}}{\sqrt{3}} \\
s \frac{\sqrt{60}}{\sqrt{10}} \\
t \frac{\sqrt{22}}{\sqrt{11}} 
\]

2. Square each of the surds below (Rule 3).

   \[
   a \sqrt{16} \\
b \sqrt{9} \\
c \sqrt{1} \\
d \sqrt{100} \\
e \sqrt{5} \\
f \sqrt{8} \\
g \sqrt{15} \\
h \sqrt{73} \\
i 2 \sqrt{2} \\
j 3 \sqrt{5} \\
k 2 \sqrt{3} \\
l 5 \sqrt{3} \\
m 7 \sqrt{3} \\
n 2 \sqrt{7} \\
o 9 \sqrt{11} \\
p 6 \sqrt{5} \\
q 10 \sqrt{10} \\
r 9 \sqrt{20} \\
s 6 \sqrt{50} \\
t 15 \sqrt{15} 
\]

3. Simplify each of these surds.

   \[
   a \sqrt{8} \\
b \sqrt{20} \\
c \sqrt{12} \\
d \sqrt{50} \\
e \sqrt{24} \\
f \sqrt{32} \\
g \sqrt{45} \\
h \sqrt{54} \\
i \sqrt{28} \\
j \sqrt{90} \\
k \sqrt{56} \\
l \sqrt{63} \\
m \sqrt{44} \\
n \sqrt{52} \\
o \sqrt{108} \\
p \sqrt{40} \\
q \sqrt{99} \\
r \sqrt{60} \\
s \sqrt{96} \\
t \sqrt{76} \\
u \sqrt{68} \\
v \sqrt{126} \\
w \sqrt{200} \\
x \sqrt{162} 
\]
4 Simplify each surd and then, taking \( \sqrt{2} = 1.41 \) and \( \sqrt{3} = 1.73 \), give a decimal approximation correct to 1 decimal place.

\[
\begin{align*}
a & \sqrt{18} & b & \sqrt{27} & c & \sqrt{8} & d & \sqrt{12} \\
e & \sqrt{32} & f & \sqrt{48} & g & \sqrt{50} & h & \sqrt{162}
\end{align*}
\]

5 Write each surd below in its simplest form.

\[
\begin{align*}
a & 2\sqrt{12} & b & 3\sqrt{8} & c & 2\sqrt{50} & d & 4\sqrt{18} \\
e & 5\sqrt{20} & f & 2\sqrt{75} & g & 10\sqrt{27} & h & 3\sqrt{56} \\
i & 2\sqrt{125} & j & 4\sqrt{45} & k & 3\sqrt{24} & l & 2\sqrt{54} \\
m & 7\sqrt{56} & n & 3\sqrt{72} & o & 3\sqrt{44} & p & 5\sqrt{90} \\
q & 6\sqrt{200} & r & 5\sqrt{98} & s & 9\sqrt{108} & t & 5\sqrt{68}
\end{align*}
\]

6 Simplified surds can be written as an entire root by reversing the above process. For example, \( 4\sqrt{3} = \sqrt{16} \times \sqrt{3} = \sqrt{48} \). Express the following as entire square roots.

\[
\begin{align*}
a & 2\sqrt{3} & b & 3\sqrt{2} & c & 2\sqrt{5} & d & 3\sqrt{6} \\
e & 4\sqrt{2} & f & 5\sqrt{3} & g & 3\sqrt{7} & h & 5\sqrt{2} \\
i & 6\sqrt{2} & j & 5\sqrt{6} & k & 3\sqrt{10} & l & 4\sqrt{7} \\
m & 6\sqrt{7} & n & 5\sqrt{10} & o & 7\sqrt{2} & p & 10\sqrt{2} \\
q & 9\sqrt{3} & r & 8\sqrt{4} & s & 7\sqrt{9} & t & 12\sqrt{3}
\end{align*}
\]

5:08 | Addition and Subtraction of Surds

<table>
<thead>
<tr>
<th>1 ( \sqrt{12} )</th>
<th>2 ( \sqrt{20} )</th>
<th>3 ( \sqrt{32} )</th>
<th>4 ( \sqrt{50} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{4} + \sqrt{9} )</td>
<td>( \sqrt{16} - \sqrt{4} )</td>
<td>( \sqrt{25} + \sqrt{36} )</td>
<td>( \sqrt{4} + \sqrt{9} ) (to 1 decimal place)</td>
</tr>
</tbody>
</table>

As can be seen from the Prep Quiz, if \( x \) and \( y \) are two positive numbers,

\( \sqrt{x} + \sqrt{y} \) does not equal \( \sqrt{x+y} \) and \n
\( \sqrt{x} - \sqrt{y} \) does not equal \( \sqrt{x-y} \)

For example:

\( \sqrt{5} + \sqrt{8} \) does not equal \( \sqrt{13} \).
\( \sqrt{10} - \sqrt{7} \) does not equal \( \sqrt{3} \).
Only 'like' surds can be added or subtracted. For example:

\[2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}\] and \[5\sqrt{2} - 4\sqrt{2} = \sqrt{2}\]

However, we can only tell whether surds are like or unlike if each is expressed in its simplest form.

Examine the following examples.

### worked examples

Simplify each of the following.

1. \[4\sqrt{3} + 7\sqrt{3} - 2\sqrt{3} = 9\sqrt{3}\]
2. \[8\sqrt{2} + \sqrt{5} - \sqrt{2} + 2\sqrt{5} = 7\sqrt{2} + 3\sqrt{5}\]
3. \[\sqrt{8} + \sqrt{18} = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}\]
4. \[\sqrt{75} + \sqrt{27} - 2\sqrt{3} = 5\sqrt{3} + 3\sqrt{3} - 2\sqrt{3} = 6\sqrt{3}\]
5. \[2\sqrt{12} + 3\sqrt{48} = 2(2\sqrt{3}) + 3(4\sqrt{3}) = 4\sqrt{3} + 12\sqrt{3} = 16\sqrt{3}\]
6. \[2\sqrt{15} - \sqrt{45} + 2\sqrt{20} = 2(\sqrt{3}) - 3\sqrt{5} + 2(2\sqrt{5}) = 10\sqrt{3} - 3\sqrt{5} + 4\sqrt{5} = 10\sqrt{3} + \sqrt{5}\]

### Exercise 5.08

1. Simplify:
   a. \[3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}\]
   b. \[4\sqrt{3} + 7\sqrt{3} = 11\sqrt{3}\]
   c. \[5\sqrt{6} + 2\sqrt{6} = 7\sqrt{6}\]
   d. \[10\sqrt{3} - 7\sqrt{3} = 3\sqrt{3}\]
   e. \[9\sqrt{6} - 6\sqrt{5} = 3\sqrt{6} - 3\sqrt{5}\]
   f. \[4\sqrt{2} - 3\sqrt{2} = \sqrt{2}\]
   g. \[\sqrt{7} + 4\sqrt{7} = 5\sqrt{7}\]
   h. \[5\sqrt{3} + \sqrt{3} = 6\sqrt{3}\]
   i. \[9\sqrt{6} - \sqrt{6} = 8\sqrt{6}\]
   j. \[9\sqrt{5} + 2\sqrt{5} + 3\sqrt{5} = 14\sqrt{5}\]
   k. \[4\sqrt{10} + 7\sqrt{10} - 2\sqrt{10} = 9\sqrt{10}\]
   l. \[4\sqrt{3} - 3\sqrt{3} + 5\sqrt{3} = 6\sqrt{3}\]
   m. \[10\sqrt{2} - 2\sqrt{2} - 5\sqrt{2} = 3\sqrt{2}\]
   n. \[\sqrt{3} + 7\sqrt{3} - 5\sqrt{3} = 3\sqrt{3}\]
   o. \[2\sqrt{7} + 3\sqrt{7} - 5\sqrt{7} = 0\sqrt{7}\]

2. Simplify by collecting like surds.
   a. \[2\sqrt{5} + 3\sqrt{7} + 4\sqrt{5} = 5\sqrt{5} + 3\sqrt{7}\]
   b. \[\sqrt{7} + 3\sqrt{7} + 2\sqrt{5} = 4\sqrt{7} + 2\sqrt{5}\]
   c. \[2\sqrt{3} + 3\sqrt{5} + 2\sqrt{3} + 5\sqrt{5} = 4\sqrt{3} + 8\sqrt{5}\]
   d. \[9\sqrt{5} + 2\sqrt{2} - 7\sqrt{3} + 3\sqrt{2} = 6\sqrt{10} + 5\sqrt{7} - 5\sqrt{10} - \sqrt{7}\]
   e. \[3\sqrt{3} + 6\sqrt{5} - \sqrt{3} - 4\sqrt{5} = 5\sqrt{3} + 2\sqrt{5}\]
   f. \[6\sqrt{11} - 2\sqrt{7} + 2\sqrt{11} + 5\sqrt{7} = 9\sqrt{2} + 3\sqrt{3} + 9\sqrt{3} - 8\sqrt{2}\]
   g. \[5\sqrt{2} + 4\sqrt{5} - 3\sqrt{5} - 6\sqrt{2} = \sqrt{2} + \sqrt{5}\]
   h. \[10\sqrt{7} - 2\sqrt{5} - 8\sqrt{7} - 3\sqrt{5} = \sqrt{7} - 5\sqrt{5}\]
3 Simplify completely:

- a \( \sqrt{8} + \sqrt{2} \)
- b \( \sqrt{12} + 2\sqrt{3} \)
- c \( 2\sqrt{2} + \sqrt{18} \)
- d \( 2\sqrt{5} + \sqrt{20} \)
- e \( \sqrt{27} + 2\sqrt{3} \)
- f \( 3\sqrt{6} + 2\sqrt{24} \)
- g \( 2\sqrt{8} - \sqrt{2} \)
- h \( 3\sqrt{5} - \sqrt{20} \)
- i \( \sqrt{32} - 3\sqrt{2} \)
- j \( \sqrt{18} + \sqrt{32} \)
- k \( \sqrt{20} + \sqrt{45} \)
- l \( 2\sqrt{27} - \sqrt{48} \)
- m \( \sqrt{75} - 2\sqrt{12} \)
- n \( \sqrt{98} + 3\sqrt{50} \)
- o \( 3\sqrt{50} + 2\sqrt{32} \)
- p \( 5\sqrt{28} + 2\sqrt{63} \)
- q \( 4\sqrt{45} - 2\sqrt{20} \)
- r \( 2\sqrt{75} - 3\sqrt{48} \)
- s \( 3\sqrt{8} + \sqrt{18} - 3\sqrt{2} \)
- t \( 2\sqrt{5} + \sqrt{45} - \sqrt{20} \)
- u \( 2\sqrt{8} - \sqrt{3} \)
- v \( 3\sqrt{5} + 2\sqrt{12} \)
- w \( 3\sqrt{27} - \sqrt{48} \)
- x \( 5\sqrt{28} + 2\sqrt{63} \)

4 Simplify:

- a \( 2\sqrt{8} - \sqrt{18} + 3\sqrt{2} \)
- d \( 5\sqrt{7} - \sqrt{63} + 2\sqrt{28} \)
- g \( 9\sqrt{8} + 3\sqrt{12} - \sqrt{27} \)
- b \( 2\sqrt{5} + \sqrt{45} - \sqrt{20} \)
- e \( 5\sqrt{3} + \sqrt{50} - \sqrt{12} \)
- h \( 5\sqrt{18} + \sqrt{72} - \sqrt{75} \)
- c \( \sqrt{27} + 2\sqrt{48} - 5\sqrt{3} \)
- f \( 2\sqrt{45} + \sqrt{20} + 3\sqrt{32} \)

---

5:09 | Multiplication and Division of Surds

**worked examples**

Simplify the following.

1. \( \sqrt{7} \times \sqrt{3} \)
2. \( 3\sqrt{5} \times 5\sqrt{2} \)
3. \( 5\sqrt{8} \times 3\sqrt{6} \)
4. \( \sqrt{96} + \sqrt{12} \)
5. \( \frac{\sqrt{18}}{\sqrt{12}} \times 4\sqrt{3} \)
6. \( \sqrt{3} \times \sqrt{5} = \sqrt{3 \times 5} \)

**Solutions**

1. \( \sqrt{7} \times \sqrt{3} = \sqrt{21} \)
2. \( 3\sqrt{2} \times 5\sqrt{2} = 3 \times 5 \times \sqrt{2} \times \sqrt{2} \)
3. \( 5\sqrt{8} \times 3\sqrt{6} \)
4. \( \sqrt{96} + \sqrt{12} = \sqrt{96 + 12} \)
5. \( \frac{\sqrt{18}}{\sqrt{12}} \times 4\sqrt{3} = \frac{4\sqrt{3} \times 3\sqrt{2}}{2\sqrt{3}} \)
6. \( \sqrt{3} \times \sqrt{5} = \sqrt{3 \times 5} \)

\[ \frac{12\sqrt{6}}{2\sqrt{3}} = 6\sqrt{2} \]

**At this point we could cancel like this:**

\[ \frac{2\sqrt{3} \times 1\sqrt{5}}{12\sqrt{2}} = 6\sqrt{2} \]
Exercise 5:09

1. Simplify these products:
   a. \(\sqrt{2} \times \sqrt{3}\)  
   b. \(\sqrt{5} \times \sqrt{7}\)  
   c. \(\sqrt{3} \times \sqrt{11}\)  
   d. \(\sqrt{2} \times \sqrt{2}\)  
   e. \(\sqrt{3} \times \sqrt{7}\)  
   f. \(\sqrt{10} \times \sqrt{7}\)  
   g. \(\sqrt{2} \times \sqrt{8}\)  
   h. \(\sqrt{3} \times \sqrt{12}\)  
   i. \(\sqrt{5} \times \sqrt{20}\)  
   j. \(\sqrt{3} \times \sqrt{6}\)  
   k. \(\sqrt{5} \times \sqrt{10}\)  
   l. \(\sqrt{2} \times \sqrt{10}\)  
   m. \(2\sqrt{3} \times \sqrt{5}\)  
   n. \(3\sqrt{2} \times 4\sqrt{5}\)  
   o. \(7\sqrt{2} \times 2\sqrt{3}\)  
   p. \(5\sqrt{2} \times 2\sqrt{2}\)  
   q. \(3\sqrt{3} \times 2\sqrt{3}\)  
   r. \(4\sqrt{5} \times \sqrt{3}\)  
   s. \(4\sqrt{2} \times 3\sqrt{10}\)  
   t. \(2\sqrt{6} \times \sqrt{8}\)  
   u. \(4\sqrt{3} \times 2\sqrt{15}\)  
   v. \(2\sqrt{x} \times 3\sqrt{x}\)  
   w. \(\sqrt{8x} \times \sqrt{2}\)  
   x. \(a\sqrt{x} \times a\sqrt{x}\)

2. Simplify:
   a. \(\sqrt{10} + \sqrt{2}\)  
   b. \(\sqrt{12} + \sqrt{4}\)  
   c. \(\sqrt{6} + \sqrt{3}\)  
   d. \(\sqrt{27} + \sqrt{3}\)  
   e. \(\sqrt{32} + \sqrt{8}\)  
   f. \(\sqrt{45} + \sqrt{5}\)  
   g. \(5\sqrt{2} + \sqrt{2}\)  
   h. \(6\sqrt{5} + \sqrt{5}\)  
   i. \(10\sqrt{3} + 5\sqrt{3}\)  
   j. \(16\sqrt{8} + 8\)  
   k. \(30\sqrt{5} + 10\)  
   l. \(24\sqrt{7} + 24\)  
   m. \(12\sqrt{10} + 2\sqrt{5}\)  
   n. \(9\sqrt{12} + 3\sqrt{6}\)  
   o. \(10\sqrt{15} + 5\sqrt{5}\)  
   p. \(\sqrt{20} + 2\)  
   q. \(\sqrt{75} + 5\)  
   r. \(5\sqrt{8} + 10\)  
   s. \(\sqrt{2x} + \sqrt{2}\)  
   t. \(\sqrt{5a} + \sqrt{a}\)  
   u. \(\sqrt{20p} + 2\sqrt{p}\)

3. Simplify fully:
   a. \(\frac{2\sqrt{3} \times 2\sqrt{6}}{4}\)  
   b. \(\frac{4\sqrt{5} \times 2\sqrt{6}}{\sqrt{10}}\)  
   c. \(\frac{2\sqrt{5} \times 3\sqrt{8}}{6\sqrt{20}}\)  
   d. \(\frac{\sqrt{15} \times \sqrt{3}}{3\sqrt{5}}\)  
   e. \(\frac{2\sqrt{3} \times \sqrt{6}}{\sqrt{12}}\)  
   f. \(\frac{3\sqrt{7} \times 2\sqrt{6}}{\sqrt{21}}\)  
   g. \(\frac{2\sqrt{6} \times 5\sqrt{5}}{10\sqrt{15}}\)  
   h. \(\frac{6\sqrt{2} \times \sqrt{6}}{4\sqrt{3}}\)  
   i. \(\frac{\sqrt{12} \times \sqrt{27}}{\sqrt{8} \times 2\sqrt{6}}\)

4. Expand and simplify:
   a. \(\sqrt{2}(\sqrt{3} + \sqrt{2})\)  
   b. \(\sqrt{5}(\sqrt{3} + \sqrt{2})\)  
   c. \(\sqrt{7}(2\sqrt{7} - \sqrt{2})\)  
   d. \(\sqrt{3}(5 - \sqrt{3})\)  
   e. \(\sqrt{2}(2\sqrt{3} - 1)\)  
   f. \(\sqrt{10}(5\sqrt{2} - 4)\)  
   g. \(2\sqrt{2}(\sqrt{2} + 1)\)  
   h. \(3\sqrt{5}(\sqrt{5} + 2)\)  
   i. \(4\sqrt{3}(\sqrt{2} - \sqrt{3})\)  
   j. \(3\sqrt{6}(\sqrt{5} + \sqrt{6})\)  
   k. \(2\sqrt{7}(\sqrt{7} - \sqrt{2})\)  
   l. \(\sqrt{3}(7 - 3\sqrt{3})\)  
   m. \(2\sqrt{2}(\sqrt{3} + 2\sqrt{2})\)  
   n. \(4\sqrt{5}(\sqrt{2} - 2\sqrt{5})\)  
   o. \(5\sqrt{6}(2\sqrt{6} - 3\sqrt{2})\)  
   p. \(\sqrt{a}(\sqrt{a} + 1)\)  
   q. \(\sqrt{x}(2\sqrt{x} + 3)\)  
   r. \(2\sqrt{y}(3\sqrt{y} + 2\sqrt{x})\)

Foundation Worksheet 5:09
Multiplication and division of surds
1. Simplify:
   a. \(\sqrt{7} \times \sqrt{5}\)  
   b. \(2\sqrt{3} \times 3\sqrt{2}\)  
   c. \(\sqrt{10} \times 2\sqrt{7}\)
Investigation 5:09 | Iteration to find square roots

Iteration is the repetition of a process. We can use a simple process to find square roots.

Example
Find \( \sqrt{3} \) correct to 4 decimal places without using a calculator.

Step 1
Estimate \( \sqrt{3} \).
Let \( E = 1.6 \)
(We want \( E^2 \) to be close to 3.)

Step 2
Divide 3 by your estimate.
\[ \frac{3}{1.6} = 1.875 \]
Since \( 1.6 \times 1.875 = 3 \), the correct answer must lie between 1.6 and 1.875.

Step 3
Average these two numbers to get a better estimate.
\[ \frac{1.6 + 1.875}{2} = 1.7375 \]
\[ \therefore \sqrt{3} \approx 1.7375 \]

• Use 1.7375 as the new estimate and repeat the steps above (iterate).
  \[ E = 1.7375 \]
  \[ \frac{3 + 1.7375}{2} = 1.73206 \]
  \[ \therefore \sqrt{3} \approx 1.73206 \]

• If we use 1.73206 as our next estimate we get a better approximation (ie \( \sqrt{3} \approx 1.732051 \)). Since \( \sqrt{3} \) lies between 1.73206 and 1.732051 then \( \sqrt{3} = 1.7321 \) correct to 4 decimal places.

1 Use iteration to find, correct to 4 decimal places:
   a \( \sqrt{2} \)      b \( \sqrt{5} \)      c \( \sqrt{70} \)      d \( \sqrt{110} \)

2 Investigate finding the square root of a number \( n \), using iteration of the formula:
   \[ \text{New estimate} = \frac{x^2 + n}{2x} \]
   where \( x \) is your last estimate and we wish to find \( \sqrt{n} \).  

To iterate, repeat the process over and over again.

Can you find other uses for iteration?
5:10 | Binomial Products

Simplify the following:

1. \( \sqrt{3} \times \sqrt{3} \)  
2. \( \sqrt{18} \times \sqrt{2} \)  
3. \( (\sqrt{5})^2 \)  
4. \( (3\sqrt{2})^2 \)  
5. \( 3\sqrt{2} + 7\sqrt{2} \)

6. \( 9\sqrt{3} - \sqrt{3} \)  
7. \( 3\sqrt{3} - 3\sqrt{3} \)

Expand and simplify where possible:

8. \( 5(\sqrt{7} - 4) \)  
9. \( \sqrt{10}(\sqrt{10} + \sqrt{3}) \)  
10. \( 2\sqrt{3}(3\sqrt{2} - \sqrt{3}) \)

In Chapter 4, you saw how to expand a binomial product. 
\((a + b)(c + d) = a(c + d) + b(c + d)\)  
\[= ac + ad + bc + bd\]

The same procedure is used if some of the terms are not pronumerals, but surds. Examine the following examples.

### Worked Examples

Expand and simplify:

1. \( a(\sqrt{2} + 3)(\sqrt{2} - 5) \)
2. \( a(2\sqrt{3} + 5)^2 \)
3. \( a(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2}) \)

**Solutions**

1. \( a(\sqrt{2} + 3)(\sqrt{2} - 5) \)
   
   \[= \sqrt{2}(\sqrt{2} - 5) + 3(\sqrt{2} - 5)\]
   
   \[= (\sqrt{2})^2 - 5\sqrt{2} + 3\sqrt{2} - 15\]
   
   \[= 2 - 2\sqrt{2} - 15\]
   
   \[= -13 - 2\sqrt{2}\]

2. \( a(2\sqrt{3} + 5)^2 \)
   
   \[= (2\sqrt{3})^2 + 2 \times 2\sqrt{3} \times 5 + (5)^2\]
   
   \[= 12 + 20\sqrt{3} + 25\]
   
   \[= 37 + 20\sqrt{3}\]

3. \( a(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2}) \)
   
   \[= (\sqrt{5})^2 - (\sqrt{2})^2\]
   
   \[= 5 - 2\]
   
   \[= 3\]

\[\text{Remember!} \]
\( (a + b)(a - b) = a^2 - b^2 \)

\[\text{Remember!} \]
\( (a + b)^2 = a^2 + 2ab + b^2 \)
\( (a - b)^2 = a^2 - 2ab + b^2 \)

These are 'perfect squares'.

These give 'the difference of two squares'.

I remember! (4:06)

These are 'perfect squares'.

I remember! (4:07A)

These give 'the difference of two squares'.

I remember! (4:07B)
Exercise 5:10

Expand and simplify the following:

1. a \((\sqrt{2} + 3)(\sqrt{2} + 1)\)  
b \((\sqrt{3} + 5)(\sqrt{3} - 1)\)  
c \((\sqrt{7} - 2)(\sqrt{7} - 5)\)  
d \((\sqrt{2} + \sqrt{3})(\sqrt{2} + \sqrt{5})\)  
e \((\sqrt{7} - \sqrt{2})(\sqrt{5} - \sqrt{2})\)  
f \((\sqrt{10} + \sqrt{2})(\sqrt{5} + \sqrt{3})\)  
g \((5 + \sqrt{2})(\sqrt{3} + \sqrt{2})\)  
h \((2 - \sqrt{3})(2 - \sqrt{5})\)  
i \((\sqrt{6} - \sqrt{2})(5 - \sqrt{6})\)  
j \((\sqrt{2} + \sqrt{3})(2\sqrt{2} + 1)\)  
k \((\sqrt{3} + 2\sqrt{5})(\sqrt{3} + \sqrt{5})\)  
l \((2\sqrt{3} + \sqrt{2})(\sqrt{3} + 2\sqrt{2})\)  
m \((5\sqrt{2} - \sqrt{7})(\sqrt{7} - 2\sqrt{2})\)  
n \((2\sqrt{5} - \sqrt{3})(5\sqrt{3} + \sqrt{5})\)  
o \((\sqrt{7} - 5\sqrt{2})(\sqrt{2} - 5\sqrt{7})\)  
p \((5\sqrt{3} + 2\sqrt{7})(2\sqrt{3} - 5\sqrt{7})\)  
q \((10\sqrt{10} + \sqrt{7})(2\sqrt{7} + \sqrt{10})\)  
r \((5\sqrt{3} + 7\sqrt{2})(7\sqrt{3} - 2\sqrt{2})\)  
s \((9\sqrt{2} - \sqrt{3})(4\sqrt{2} + 2\sqrt{3})\)  
t \((5\sqrt{7} + 4)(2\sqrt{7} - 7)\)  
u \((6\sqrt{3} + 5)(7 - 2\sqrt{3})\)  
v \((\sqrt{x} + 3)(\sqrt{x} + 2)\)  
w \((\sqrt{m} + \sqrt{n})(2\sqrt{m} + \sqrt{n})\)  
x \((3\sqrt{a} - 2\sqrt{b})(2\sqrt{a} + 3\sqrt{b})\)

2. a \((\sqrt{2} + 1)^2\)  
b \((\sqrt{3} - 5)^2\)  
c \((\sqrt{5} + 2)^2\)  
d \((\sqrt{3} + \sqrt{2})^2\)  
e \((\sqrt{5} - \sqrt{2})^2\)  
f \((\sqrt{3} + \sqrt{10})^2\)  
g \((2\sqrt{3} + 1)^2\)  
h \((3\sqrt{2} - 4)^2\)  
j \((3\sqrt{5} + \sqrt{10})^2\)  
k \((3\sqrt{5} + \sqrt{10})^2\)  
m \((5\sqrt{3} + 2\sqrt{2})^2\)  
n \((7\sqrt{3} - 2\sqrt{5})^2\)  
o \((2\sqrt{m} + 5)^2\)  
p \((2\sqrt{m} + 5)^2\)  
q \((2\sqrt{n} + 5)^2\)  
r \((3\sqrt{a} - 2\sqrt{b})^2\)

3. a \((\sqrt{2} + 1)(\sqrt{2} - 1)\)  
b \((5 + \sqrt{3})(5 - \sqrt{3})\)  
c \((\sqrt{10} - 7)(\sqrt{10} + 7)\)  
d \((4 - \sqrt{2})(4 + \sqrt{2})\)  
e \((\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})\)  
f \((\sqrt{7} - \sqrt{5})(\sqrt{7} + \sqrt{5})\)  
g \((\sqrt{10} - \sqrt{8})(\sqrt{10} + \sqrt{8})\)  
h \((\sqrt{11} + \sqrt{7})(\sqrt{11} - \sqrt{7})\)  
i \((2\sqrt{3} - 5)(2\sqrt{3} + 5)\)  
j \((6 - 3\sqrt{2})(6 + 3\sqrt{2})\)  
k \((\sqrt{7} + 2\sqrt{2})(\sqrt{7} - 2\sqrt{2})\)  
l \((3\sqrt{5} - \sqrt{3})(3\sqrt{5} + \sqrt{3})\)  
m \((5\sqrt{2} - 2\sqrt{5})(5\sqrt{2} + 2\sqrt{5})\)  
n \((2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2})\)  
o \((\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})\)  
p \((2\sqrt{a} + 3\sqrt{b})(2\sqrt{a} - 3\sqrt{b})\)

Important notice!
The two binomials in each part of question 3 are said to be conjugate surds.
Note that when a binomial surd is multiplied by its ‘conjugate’, the answer is always a rational number.
Rationalising the Denominator

If a fraction has a surd (i.e., an irrational number) in its denominator, we generally rewrite the fraction with a ‘rational’ denominator by using the method shown below.

Simplify the following:

1. $\sqrt{5} \times \sqrt{5}$  
2. $\sqrt{10} \times 10$  
3. $\sqrt{2} \times \sqrt{3}$  
4. $5 \times \sqrt{2} \times \sqrt{2}$  
5. $2 \times \sqrt{6} \times \sqrt{6}$  
6. $(\sqrt{2} + 1)(\sqrt{2} - 1)$  
7. $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$  
8. $(5 - \sqrt{2})(5 + \sqrt{2})$  
9. $(2\sqrt{3} - \sqrt{2})(2\sqrt{3} + \sqrt{2})$  
10. $(5\sqrt{2} - 3\sqrt{3})(5\sqrt{2} + 3\sqrt{3})$

Rewrite with rational denominators:

1. $\frac{3}{\sqrt{3}} = \frac{3 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{3\sqrt{3}}{3}$
2. $\frac{1}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$
3. $\frac{\sqrt{5}}{\sqrt{12}} = \frac{\sqrt{5} \times \sqrt{3}}{2 \sqrt{3} \times \sqrt{3}} = \frac{\sqrt{15}}{2 \times 3}$
4. $\frac{2 + \sqrt{3}}{2\sqrt{3}} = \frac{2 + \sqrt{3} \times \sqrt{3}}{2\sqrt{3} \times \sqrt{3}} = \frac{2 + 3}{2 \times 3} = \frac{5}{6}$

Note: Multiplying by $\frac{\sqrt{3}}{\sqrt{3}}$ is the same as multiplying by 1.

Exercise 5:11

Rationalise the denominator for each of the following:

a. $\frac{1}{\sqrt{2}}$  
b. $\frac{1}{\sqrt{3}}$  
c. $\frac{2}{\sqrt{3}}$  
d. $\frac{5}{\sqrt{10}}$  
e. $\frac{3}{\sqrt{2}}$  
f. $\frac{6}{\sqrt{3}}$  
g. $\frac{10}{\sqrt{5}}$  
h. $\frac{2}{\sqrt{11}}$  
i. $\frac{\sqrt{2}}{\sqrt{3}}$  
j. $\frac{\sqrt{3}}{\sqrt{5}}$  
k. $\frac{\sqrt{5}}{\sqrt{10}}$  
l. $\frac{\sqrt{3}}{\sqrt{15}}$

m. $\frac{1}{2\sqrt{2}}$  
n. $\frac{2}{5\sqrt{3}}$  
o. $\frac{7}{2\sqrt{5}}$  
p. $\frac{10}{2\sqrt{3}}$  
q. $\frac{\sqrt{6}}{2\sqrt{3}}$  
r. $\frac{\sqrt{5}}{5\sqrt{2}}$

s. $\frac{2\sqrt{3}}{3\sqrt{2}}$  
t. $\frac{5\sqrt{7}}{3\sqrt{5}}$  
u. $\frac{2 + \sqrt{3}}{\sqrt{3}}$  
w. $\frac{1 + \sqrt{5}}{\sqrt{2}}$  
x. $\frac{\sqrt{10} - \sqrt{5}}{5\sqrt{10}}$
2 Evaluate each fraction correct to 3 significant figures (using your calculator). Then rationalise
the denominator and evaluate the fraction again. Compare this answer with your first
calculation.

\[
\begin{align*}
a & = \frac{2}{\sqrt{5}} \\
b & = \frac{3}{\sqrt{7}} \\
c & = \frac{\sqrt{3}}{2\sqrt{2}} \\
d & = \frac{\sqrt{7}}{3\sqrt{5}}
\end{align*}
\]

3 Rationalise each denominator, then express as a single fraction.

\[
\begin{align*}
a & = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \\
b & = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{2}} \\
c & = \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{5}} \\
d & = \frac{2}{\sqrt{10}} - \frac{3}{\sqrt{5}} \\
e & = \frac{3}{\sqrt{8}} + \frac{5}{\sqrt{2}} \\
f & = \frac{2}{2\sqrt{3}} - \frac{1}{3\sqrt{2}} \\
g & = \frac{2}{5\sqrt{2}} + \frac{5}{\sqrt{10}} \\
h & = \frac{\sqrt{5} + \sqrt{7}}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{5}} \\
i & = \frac{\sqrt{2} - \sqrt{3}}{\sqrt{5}} \\
j & = \frac{1}{x} = 1 \\
k & = \frac{14}{x} = 1 \\
l & = \frac{2x - 5}{6x} = 7
\end{align*}
\]

---

### Fun Spot 5:11 | What do Eskimos sing at birthday parties?

Answer each question and put the letter
for that question in the box above the
correct answer.

<table>
<thead>
<tr>
<th>A</th>
<th>E</th>
<th>O</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{x}{2} = 5)</td>
<td>(\frac{x}{3} = 3)</td>
<td>(\frac{x}{10} = 1.86)</td>
<td>(0.560 = \frac{x}{5})</td>
</tr>
</tbody>
</table>

Simplify:

\[
\begin{align*}
L & = 5x - x \\
E & = 2x \times 3y \\
F & = (2x^3)^3 \\
E & = \sqrt{169} \\
I & = \sqrt{50} \\
O & = 5\sqrt{2} \times 3\sqrt{2} \\
R & = \sqrt{32} + \sqrt{2}
\end{align*}
\]

Solve:

\[
\begin{align*}
W & = 5x + 3 = 21 \\
Y & = \frac{14}{x} = 1 \\
Z & = 2x - 5 = 7 - 6x
\end{align*}
\]
Challenge 5:11 | Rationalising binomial denominators

**Examples**

Rationalise the denominators for each expression.

1. \( \frac{5}{5 - \sqrt{2}} \)
2. \( \frac{1}{\sqrt{3} + \sqrt{2}} \)
3. \( \frac{2 \sqrt{3} + \sqrt{5}}{2 \sqrt{3} - \sqrt{5}} \)

**Solutions**

1. \( \frac{5}{5 - \sqrt{2}} = \frac{5(5 + \sqrt{2})}{5 - \sqrt{2} \times 5 + \sqrt{2}} = \frac{25 + 5 \sqrt{2}}{25 - 2} = \frac{25 + 5 \sqrt{2}}{23} \)

2. \( \frac{1}{\sqrt{3} + \sqrt{2}} = \frac{1 \times \sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2} \times \sqrt{3} - \sqrt{2}} = \frac{\sqrt{3} - \sqrt{2}}{3 - 2} = \sqrt{3} - \sqrt{2} \)

3. \( \frac{2 \sqrt{3} + \sqrt{5}}{2 \sqrt{3} - \sqrt{5}} = \frac{2 \sqrt{3} + \sqrt{5} \times 2 \sqrt{3} + \sqrt{5}}{2 \sqrt{3} - \sqrt{5} \times 2 \sqrt{3} + \sqrt{5}} = \frac{(2 \sqrt{3} + \sqrt{5})^2}{12 - 5} = \frac{17 + 4 \sqrt{15}}{7} \)

**Note:**

The product of a binomial surd and its conjugate is always rational.

Now try these exercises!

1. Express with a rational denominator.
   a. \( \frac{1}{1 + \sqrt{2}} \)
   b. \( \frac{1}{\sqrt{3} - 1} \)
   c. \( \frac{1}{\sqrt{7} - \sqrt{5}} \)
   d. \( \frac{1}{\sqrt{10} + \sqrt{2}} \)
   e. \( \frac{3}{\sqrt{3} + 2} \)
   f. \( \frac{5}{5 - \sqrt{2}} \)
   g. \( \frac{10}{\sqrt{5} - \sqrt{2}} \)
   h. \( \frac{12}{\sqrt{7} - \sqrt{3}} \)
   i. \( \frac{1}{2 \sqrt{3} + 5} \)
   j. \( \frac{2}{5 - 2 \sqrt{2}} \)
   k. \( \frac{3}{3 \sqrt{2} + 2 \sqrt{3}} \)
   l. \( \frac{1}{4 \sqrt{3} - 3 \sqrt{2}} \)
   m. \( \frac{5 + \sqrt{2}}{5 - \sqrt{2}} \)
   n. \( \frac{4 + \sqrt{3}}{4 - \sqrt{3}} \)
   o. \( \frac{\sqrt{3} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} \)
   p. \( \frac{3 \sqrt{2} - \sqrt{3}}{3 \sqrt{2} + \sqrt{3}} \)

2. Rationalise each denominator, then express as a single fraction.
   a. \( \frac{1}{2 - \sqrt{3}} + \frac{1}{2 + \sqrt{3}} \)
   b. \( \frac{1}{\sqrt{5} - \sqrt{3}} - \frac{1}{\sqrt{7} + \sqrt{3}} \)
   c. \( \frac{5}{6 - \sqrt{3}} + \frac{3}{5 + \sqrt{3}} \)
Mathematical Terms 5

base
- The term which is operated on by the index.
  eg for \( x^n \), \( x \) is the base
  for \( 5^3 \), \( 5 \) is the base.

conjugate
- The binomials that multiply to give the difference of two squares are the ‘conjugate’ of each other.
  eg \((a - b)\) and \((a + b)\)
  \((\sqrt{3} + 5)\) and \((\sqrt{3} - 5)\)
  These are conjugate pairs.

exponent
- Another term for a power or index.
- Equations which involve a power are called exponential equations.
  eg \( 3^x = 27 \)

fractional indices
- Another way of writing the ‘root’ of a number or term.
- \( \frac{1}{3} \), \( \sqrt[3]{x} \)
  \( x^3 = \frac{1}{\sqrt[3]{x}}, x^n = \sqrt[n]{x} \)

index
- A number indicating how many of a base term need to be multiplied together.
  eg for \( x^n \), \( n \) is the index
  \( x^n = x \times x \times x \times \ldots \times x \)
  \( n \) factors
- The plural of index is indices.

irrational numbers
- Numbers that cannot be expressed in the form \( \frac{a}{b} \) where \( a \) and \( b \) are integers.
- They cannot be given an exact decimal value.
  eg \( \pi, \sqrt{3}, 3\sqrt{2} + 1 \)

negative indices
- Indicate the reciprocal of a term.
  eg \( x^{-1} = \frac{1}{x}, x^{-n} = \frac{1}{x^n} \)
  ie \( 5^{-1} = \frac{1}{5}, 2^{-3} = \frac{1}{2^3} = \frac{1}{8} \)

power
- Another term for an index or exponent.

rational numbers
- Numbers that can be written in the form \( \frac{a}{b} \) where \( a \) and \( b \) are integers (\( b \neq 0 \)).
- They can be expressed as a terminating or repeating decimal.
  eg integers, fractions, percentages

real numbers
- The combination of rational and irrational numbers.

scientific (standard) notation
- A useful way to write very big or very small numbers.
- Numbers are written as the product of a number between 1 and 10 and a power of 10.
  eg \( 76 000 000 = 7.6 \times 10^7 \)
  \( 0.000 0054 = 5.4 \times 10^{-6} \)

surds
- Numerical expressions that involve irrational roots.
  eg \( \sqrt{3}, \frac{1}{\sqrt{5}}, 2\sqrt{7} + 5 \)

zero index
- A term or number with a zero index is equal to 1.
  eg \( x^0 = 1, 4^0 = 1 \)
# Diagnostic Test 5: Indices and Surds

- These questions reflect the important skills introduced in this chapter.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

<table>
<thead>
<tr>
<th></th>
<th>Express in index form:</th>
<th></th>
<th>Evaluate:</th>
<th></th>
<th>Simplify:</th>
<th></th>
<th>Simplify, writing answers without negative indices:</th>
<th></th>
<th>Simplify:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a  $3 \times 3 \times 3 \times 3$</td>
<td>b</td>
<td>$5 \times 5$</td>
<td>c</td>
<td>$m \times m \times m$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a  $3^2$</td>
<td>b</td>
<td>$2^4$</td>
<td>c</td>
<td>$10^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>a  $3^2 \times 3^5$</td>
<td>b</td>
<td>$x^3 \times x^2$</td>
<td>c</td>
<td>$6m^2n \times mn^4$</td>
<td></td>
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</tr>
<tr>
<td>4</td>
<td>a  $x^2 + x^2$</td>
<td>b</td>
<td>$15a^5 + 3a^2$</td>
<td>c</td>
<td>$20a^3b^2 + 10ab$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>a  $(a^2)^2$</td>
<td>b</td>
<td>$(x^3)^4$</td>
<td>c</td>
<td>$(2a^4)^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>a  $7^0$</td>
<td>b</td>
<td>$5p^0$</td>
<td>c</td>
<td>$18x^3 + 6x^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>a  $3^{-2}$</td>
<td>b</td>
<td>$5^{-1}$</td>
<td>c</td>
<td>$(\frac{2}{3})^{-3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>a  $x^7 \times x^{-3}$</td>
<td>b</td>
<td>$6x^2 + 3x^4$</td>
<td>c</td>
<td>$(3x^{-1})^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>a  $\frac{1}{25^2}$</td>
<td>b</td>
<td>$\frac{1}{27^3}$</td>
<td>c</td>
<td>$\frac{1}{8^3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>If $x &gt; 0$ and $m &gt; 0$, simplify:</td>
<td>a</td>
<td>$3x^{\frac{1}{2}} \times 4x^{\frac{1}{2}}$</td>
<td>b</td>
<td>$(49m^6)^{\frac{1}{2}}$</td>
<td>c</td>
<td>$(8x^3)^{\frac{1}{3}}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Express in scientific notation:</td>
<td>a</td>
<td>$243$</td>
<td>b</td>
<td>$67,000$</td>
<td>c</td>
<td>$93,800,000$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Write as a basic numeral:</td>
<td>a</td>
<td>$1.3 \times 10^2$</td>
<td>b</td>
<td>$2.431 \times 10^2$</td>
<td>c</td>
<td>$4.63 \times 10^7$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Express in scientific notation:</td>
<td>a</td>
<td>$0.043$</td>
<td>b</td>
<td>$0.000,059,7$</td>
<td>c</td>
<td>$0.004$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>Write the basic numeral for:</td>
<td>a</td>
<td>$2.9 \times 10^{-2}$</td>
<td>b</td>
<td>$9.38 \times 10^{-5}$</td>
<td>c</td>
<td>$1.004 \times 10^{-3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Simplify, giving answers in scientific notation:</td>
<td>a</td>
<td>$(3.1 \times 10^8)^2$</td>
<td>b</td>
<td>$(8.4 \times 10^6) + (3.8 \times 10^7)$</td>
<td>c</td>
<td>$\sqrt{1.96 \times 10^{24}}$</td>
<td>d</td>
<td>$\sqrt{1.44 \times 10^{-6}}$</td>
<td></td>
</tr>
</tbody>
</table>
16 For each, write rational or irrational.
   a $\sqrt{50}$  b 0.4  c $\sqrt{2}$  d $\sqrt{16}$

17 Evaluate correct to 3 decimal places:
   a $\sqrt{5}$  b $\sqrt{13}$  c $\sqrt{21}$  d $\sqrt{47}$

18 Simplify each surd.
   a $\sqrt{20}$  b $\sqrt{27}$  c $3\sqrt{8}$  d $2\sqrt{75}$

19 Express each of the following as an entire square root.
   a $2\sqrt{5}$  b $3\sqrt{2}$  c $5\sqrt{7}$  d $4\sqrt{5}$

20 Simplify completely:
   a $2\sqrt{3} + 4\sqrt{3}$  b $6\sqrt{5} - \sqrt{5}$  c $\sqrt{8} - \sqrt{2}$  d $\sqrt{27} + 2\sqrt{3}$

21 Simplify:
   a $\sqrt{5} \times \sqrt{6}$  b $\sqrt{3} \times \sqrt{12}$  c $2\sqrt{3} \times \sqrt{5}$  d $3\sqrt{8} \times 2\sqrt{2}$

22 Simplify:
   a $\sqrt{12} + \sqrt{2}$  b $\sqrt{32} + \sqrt{8}$  c $5\sqrt{3} + \sqrt{3}$  d $10\sqrt{10} + 2\sqrt{5}$

23 Expand and simplify:
   a $(2 + \sqrt{5})(3 + \sqrt{5})$  b $(2\sqrt{3} + \sqrt{2})(\sqrt{3} - 3\sqrt{2})$
   c $(\sqrt{7} + \sqrt{3})^2$  d $(5 - \sqrt{3})(5 + \sqrt{3})$

24 Rationalise the denominator of:
   a $\frac{3}{\sqrt{2}}$  b $\frac{5}{\sqrt{5}}$  c $\frac{\sqrt{3} + 1}{2\sqrt{3}}$  d $\frac{5 - \sqrt{2}}{5\sqrt{2}}$

• Can you use your calculator to find the value of $2^{500}$? What is the largest power of 2 that can be calculated using your calculator?
Chapter 5 | Revision Assignment

1 Simplify, writing the answers in index form:
   a \( a^2 \times a^3 \)
   b \( 3a^2 \times 4a^3 \)
   c \( a^2b \times ab \)
   d \( 3a^2b \times 4ab^2 \)
   e \( 3^2 \times 3^3 \)
   f \( a^6 + a^3 \)
   g \( 7m^2 + m \)
   h \( 12y^6 + 3y^2 \)
   i \( 20d^6b^3 + 10a^2b^2 \)
   j \( 4^7 + 4^3 \)
   k \( (3^2)^4 \)
   l \( (x^2)^3 \)
   m \( (a^2)^2 \times a^5 \)
   n \( m^7 + (m^2)^3 \)
   o \( \frac{12x^2}{6x} \)
   p \( \frac{4a^3}{8a} \)

2 Express in simplest form:
   a \( (2x^2)^5 \)
   b \( 6x^0 \)
   c \( (5x)^3 \)
   d \( (10a)^3 \)
   e \( (4x^3)^3 + 8x^5 \)

3 Write each of the following in standard form (scientific notation):
   a \( 21,600 \)
   b \( 125 \)
   c \( 0.000007 \)
   d \( 0.000156 \)

4 Write each of the following as a basic numeral:
   a \( 8.1 \times 10^5 \)
   b \( 1.267 \times 10^3 \)
   c \( 3.5 \times 10^{-2} \)
   d \( 1.06 \times 10^{-4} \)

5 Use your calculator to evaluate:
   a \( 2^{10} \)
   b \( 3^{12} \)
   c \( 5^2 \times 6^6 \)
   d \( 7^3 \times 4^5 \)

6 Find the value of \( n \), if:
   a \( 2^n = 128 \)
   b \( 3^n = 243 \)
   c \( 10^n = 100,000,000 \)

7 Simplify and evaluate:
   a \( 3^2 \times 3^5 \)
   b \( 10^7 \times 10^4 \)
   c \( (2^4)^2 \)

8 Simplify:
   a \( \frac{m^7 \times m^6}{m^{10}} \)
   b \( (2a^3)^2 \times (3a^4)^2 \)
   c \( 8x^7 \times 9x^4 \)
   d \( 6x^6 \times 6x^3 \)

9 Noting that \( x^{\frac{3}{2}} = (x^3)^{\frac{1}{2}} \), evaluate without using a calculator:
   a \( 4^\frac{3}{2} \)
   b \( 8^\frac{2}{3} \)
   c \( 9^\frac{3}{2} \)
   d \( 1000^\frac{3}{3} \)

10 Simplify:
   a \( \frac{1}{5x^2} \times \frac{1}{4x^3} \)
   b \( 10x^2 - 5x^4 \)
   c \( (36m^4n^6)^{\frac{1}{2}} \)

11 Simplify each expression.
   a \( 5\sqrt{20} \)
   b \( \sqrt{28} + 3\sqrt{7} \)
   c \( 3\sqrt{8} - 2\sqrt{18} \)
   d \( 2\sqrt{7} \times 3\sqrt{2} \)
   e \( 2\sqrt{5} \times \sqrt{20} \)
   f \( 15\sqrt{12} + 5\sqrt{3} \)
   g \( \sqrt{3}(\sqrt{3} + \sqrt{13}) \)
   h \( (\sqrt{7} + \sqrt{2})^2 \)
   i \( (\sqrt{5} + \sqrt{6})(\sqrt{2} + \sqrt{5}) \)
   j \( \sqrt{m} \times \sqrt{n} \)
   k \( (\sqrt{m} + \sqrt{n})^2 \)
   l \( (\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n}) \)

12 Rationalise the denominator of each expression:
   a \( \frac{5}{2\sqrt{3}} \)
   b \( \frac{2\sqrt{5}}{3\sqrt{2}} \)
   c \( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \)
   d \( \frac{3}{2\sqrt{5}} - \frac{2}{3\sqrt{2}} \)

Extension

13 Rationalise the denominator of each expression.
   a \( \frac{2}{\sqrt{5} - 1} \)
   b \( \frac{1}{\sqrt{7} + \sqrt{2}} \)
   c \( \frac{\sqrt{5} - 2}{\sqrt{5} + 2} \)
   d \( \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2} - \sqrt{3}} \)

14 Simplify each expression, writing your answer with a rational denominator.
   a \( \frac{5 \sqrt{3} - 2}{\sqrt{3} - 1} \)
   b \( \frac{3 \sqrt{5} \times 2}{\sqrt{5} + 2} \)
   c \( \frac{\sqrt{7} + 2 \sqrt{3} + 1}{\sqrt{7} - 1} \)
Chapter 5 | Working Mathematically

1 Use ID Card 7 on page xix to identify:
   a 5  b 8  c 17  d 18  e 19
   f 20  g 21  h 22  i 23  j 24

2 Use ID Card 6 on page xviii to identify numbers 1 to 12.

3 a How many diagonals can be drawn from one vertex of a regular hexagon? How many vertices has a hexagon?
   b Each diagonal joins two vertices and a diagonal cannot be drawn from a vertex to the two adjacent vertices or to itself. The number of diagonals of a hexagon is $\frac{6(6 - 3)}{2}$. How many diagonals has:
      i a regular octagon?
      ii a regular decagon?
      iii a regular polygon that has 30 sides?

4 Tom was given a cheque for an amount between $31 and $32. The bank teller made a mistake and exchanged dollars and cents on the cheque. Tom took the money without examining it and gave 5 cents to his son. He now found that he had twice the value of the original cheque. If he had no money before entering the bank, what was the amount of the cheque?

5 In the decibel scale, for measuring noise, 10 decibels is a noise that is barely audible. A noise 10 times as intense is 20 decibels, and so on up to 140 decibels, which is the threshold of pain. Study the table and answer the questions below.

<table>
<thead>
<tr>
<th>Noise</th>
<th>Relative intensity</th>
<th>Decibels</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum of audible sound</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Soft wind on leaves</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Whisper at 1 metre</td>
<td>$10^2$</td>
<td>20</td>
</tr>
<tr>
<td>Bush quiet</td>
<td>$10^3$</td>
<td>30</td>
</tr>
</tbody>
</table>

a If ordinary conversation has a relative intensity of $10^6$, what is its loudness in decibels?
b If a lawn mower has a relative intensity of $10^{12}$, what is its loudness in decibels?
c By how many times is the relative intensity of the mower greater than that of conversation?
d By how many times is the relative intensity of heavy traffic (loudness 80 dB) greater than that of bush quiet?
e From the above it would appear that heavy traffic (80 dB) is four times as noisy as a whisper at 1 metre (20 dB). However, a rise of 10 dB corresponds to a doubling in the subjective loudness to the human ear. How much louder to the human ear is:
   i the average office (50 dB) than bush quiet (30 dB)?
   ii heavy traffic (80 dB) than a whisper at 1 metre (20 dB)?
   iii a rock group (110 dB) than a business office (60 dB)?